THE LINDSTROM-GESSEL-VIENNOT LEMMA AND ITS APPLICATIONS

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1. The Lemma

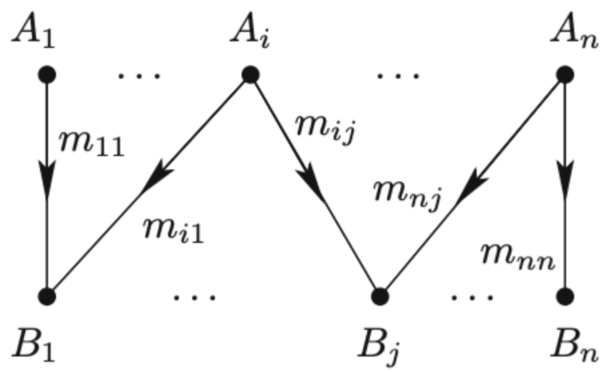
Our starting paint is a real $n \times n$ matrix, $M = (m_{ij})$. The Leibez formula gives us the determinant of M in terms of the permutations of the matrix elements. So we have

$$\det M = \sum_{\sigma} \text{sign}\sigma m_{1\sigma(1)} m_{2\sigma(2)} ... m_{n\sigma(n)}$$

Note that the sign of σ may be -1 or 1. This depends on whether the number of transpositions is even or odd.

Definition 1.1. A graph is a bipartite iff its vertex sets may be partitioned into two disjoint sets, such that every edge in the graph joins a vertex from one set to a vertex in the other set. We may say that this graph is directed/weighted if the edges have a direction/numerical weighting associated with them.

We consider a weighted directed bipartite graph. Let the vertices $A_1...A_n$ represent rows of M, and $B_1...B_n$ represent columns of M. For some $A_i \to B_j$, the weight will be represented by m_{ij}.



Let A = $\{A_1...A_n\}$ and B = $\{B_1...B_n\}$. For a given system, P_σ, the weighted (signed) sum over all vertex-disjoint path systems $A \to B$ is given by paths

$$A_1 \to B_{\sigma(1)}, ..., A_n \to B_{\sigma(n)}$$

The product of each individual weight represents the weight on the system so we have

$$w(P_\sigma) = w(A_1 \to B_{\sigma(1)})...w(A_n \to B_{\sigma(n)})$$

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Thus we find, for the matrix M,

$$\det M = \sum_{\sigma} \text{sign}\sigma w(P_\sigma)$$

Now, we show that it is possible to generalize this result from bipartite to arbitrary graphs, a result originally shown by Gessel and Viennot.

Consider a finite acyclic directed graph $G = (V, E)$ (in other words, a graph G with no directed cycles, and where there are a finite number of vertexes and edges contained in G). Including all trivial paths $A \to A$ of length 0, we find that the number of directed paths between A and B is finite.

Definition 1.2. Let $w(e)$ represent the weight of some edge e. The weight of P where $P : A \to B$ can be defined as

$$w(P) := \prod_{e \in P} w(e)$$

Note that $w(e) = 1$ in the case that the length of P is 0.

Now we again consider A = $\{A_1 ... A_n\}$ and B = $\{B_1 ... B_n\}$, where A represents the columns of a matrix $M = (m_{ij})$, and B represents the rows of M. Note that A and B need not be disjoint. We know that

$$m_{ij} := \sum_{P : A_i \to B_j} w(P)$$

As we know, the weight of the path system P (where P is from A to B) is the product of all the edges in the system. This can also be written as

$$w(P) = \prod_{i=1}^{n} w(P_i) \tag{1.1}$$

where $P = sign\sigma$ and there are n paths.

We know that a path system $P = (P_1, ..., P_n)$ is vertex-disjoint if the paths of P are pair-wise vertex disjoint.

Lemma 1.3. *Let $G = (V, E)$ be a finite weighted acyclic directed graph, $A = A_1, ..., A_n$ and $B = B_1, ..., B_n$ two n-sets of vertices, and M the path matrix from A to B. Then*

$$\det M = \sum_{P \text{ vertex-disjoint path system}} signPw(P)$$

Proof. As we know, det(M) can be written as

$$\det M = \sum_{\sigma} \text{sign}\sigma m_{1\sigma(1)} m_{2\sigma(2)} ... m_{n\sigma(n)}$$

This becomes

$$\det M = \sum_{\sigma} \text{sign}\sigma \Big(\sum_{P_1 : A_1 \to B_{\sigma(1)}} w(P_1) \Big) ... \Big(\sum_{P_n : A_n \to B_{\sigma(n)}} w(P_n) \Big)$$

From 1.1, we find that, summing over σ

$$\det M = \sum_{P} \text{sign} P w(P)$$

Let N be the set of path systems that are not disjoint. In order to prove the statement of the lemma all we need to do is prove

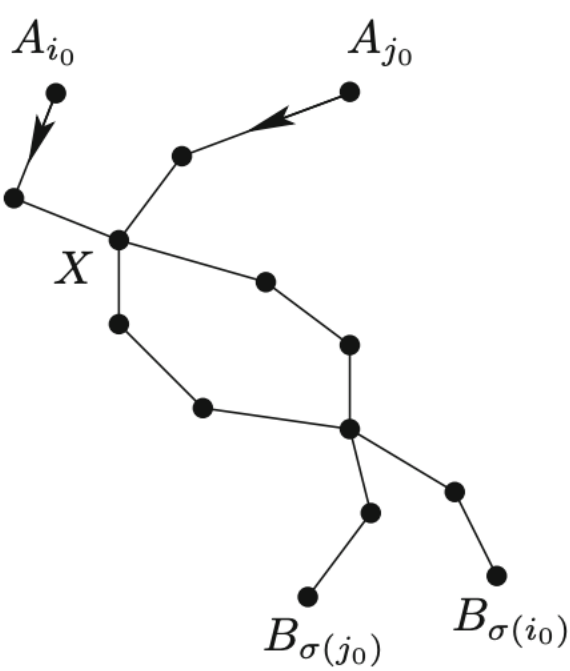
$$\sum_{P\in N} \text{sign}Pw(P) = 0$$

because this will show that non disjoint paths do not contribute to the determinant. In order to do this, we define an involution $\pi : N \to N$ with no fixed points which satisfies the following conditions

$$w(\pi P) = w(P)$$
$$\text{sign}\pi P = -\text{sign}P$$

This would satisfy the statement we need to prove, and hence, the lemma. Let us better define the involution π. For $P \in N$, with paths $P_i : A_i \to B_{\sigma(i)}$, we know that some pairs of paths will intersect (by definition).

Take i_0 to be the minimal index where P_{i0} shares a vertex with another path, and let the first of these common vertices be X. Take j_0 to be the minimal index such that P_{j0} shares this vertex X with P_{i0} and $j_0 > i_0$.



We construct a new system $\pi P = P'_1, ..., P'_n$ where when $k \neq i_0, j_0$, $P'_k = P_k$. The new path we have constructed P'_{i0} goes from A_{i0} to X along P_{i0}, and then to $B_{\sigma(j0)}$ along P_{j0}. The path P'_{j0} travels along A_{j0} to X along P_{j0}, continuing to $B_{\sigma(i0)}$ along P_{i0}.

We can see that $\pi(\pi P) = P$, given that index i_0, vertex X, and index j_0 remain the same as before.

So, when we apply π twice, we end up switching back to the original path P_i. Additionally P and πP have the same edges, so obviously $w(\pi P) = w(P)$. The new permutation σ' is given by the multiplication of σ with the transposition (i_0, j_0), so we can see that $\text{sign}\pi P = -\text{sign}P$. This can be generalized for all paths which share at least one vertex with another path (making them non-disjoint), and thus for all $P \in N$. Thus, we have proven our statements, and the lemma. Note that the graph we consider must be acyclic because the involution, π could transform a self-intersection of the path into an intersection of two distinct paths. This would break the involution argument. ■

The Lindström–Gessel–Viennot lemma can help us to derive all of the basic properties of determinants through the use of an appropriate graph.

2. Applications

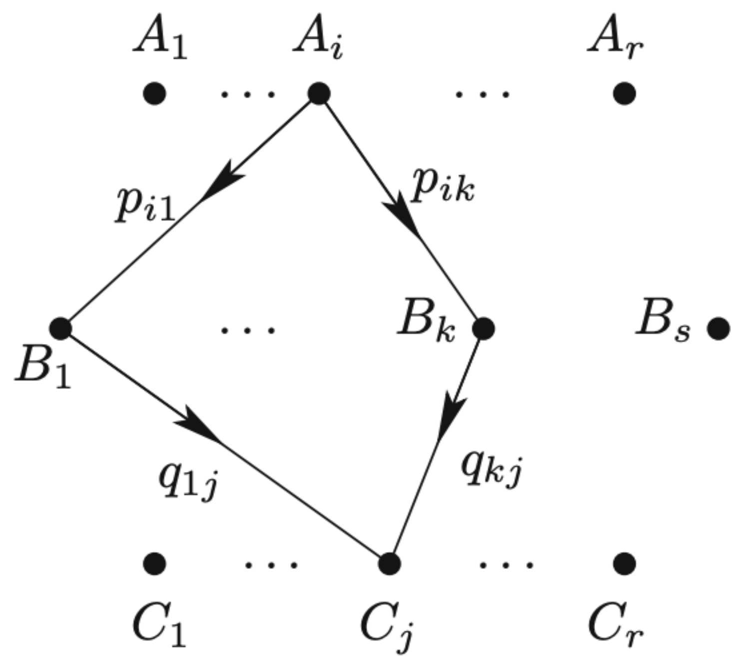
One example of the application of this lemma is in proving the Binet–Cauchy formula for the determinant of the product of two rectangular matrices.

Theorem 2.1. *If P is an $r \times s$ matrix and Q an $s \times r$ matrix, $r \leq s$, then*

$$\det PQ = \sum_{Z} (\det P_Z)(\det Q_Z),$$

where P_Z is the $r \times r$ submatrix of P with column-set Z, and Q_Z the $r \times r$ submatrix of Q with the corresponding rows Z.

Proof. We again start by considering a bipartite graph on A and B, relating to a path system P. Consider also a bipartite graph on B and C, relating to paths Q. We now link these together to create a new graph between A and C.



We see that for each m_{ij} of the path matrix from A to C is $m_{ij} = \sum_k p_{ik} q_{kj}$ so the matrix is equivalent to PQ. With vertex disjoint path systems A to C corresponding with A to Z corresponding with Z to C, the result signσT = (signσ)(sign$T$) follows from the lemma. ■

References

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[3] B. LINDSTRÖM: On the vector representation of induced matroids, Bulletin London Math. Soc. 5 (1973), 85-90.