THE BORROMEAN RINGS AND HYPERBOLIC LINKS

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1. The Borromean Rings Don't Exist

Definition 1.1. The *Borromean Rings* consists of 3 unknots that are linked, but are pairwise unlinked, or trivially linked.

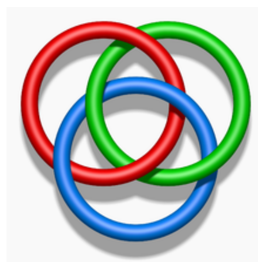

Figure 1. The Borromean Rings

A natural question that arises after seeing the Borromean rings is

Question 1.2. *Is it possible to form the Borromean rings with components that are perfect circles?*

It turns out that this is in fact impossible. The proof of this is attributed to Freedman and Skora (1987).

Theorem 1.3. *The Borromean Rings are a nontrivial link.*

We will show this using Fox n-labelings, a method of finding if a knot or link is nontrivial.

Definition 1.4. The *Fox n-labeling* of a link diagram is a labeling of each arc in a knot diagram such that at each crossing the two integers a and c of the arcs that end at the crossing and the label b of the arc of the overpass satisfy the *crossing relation*

$$a + c \equiv 2b \pmod{n}.$$

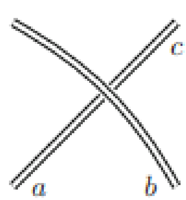

Figure 2. The Crossing Relation

Date: August 2020.

Each link diagram has n *trivial* n-labelings, where all the arcs are labeled the same (i.e. $a = b = c$ everywhere). We will only be interested in the *non-trivial labelings*, which use at least two labels for some n.

Lemma 1.5. *If two link diagrams represent the same link, then they have the same Fox n-labelings.*

Proof. Since all equivalent knots can be formed by deformations using the Reidemeister moves, it would be simplest to show that the Reidemeister moves preserve the n-labelings. Then, we can draw the Reidemeister moves

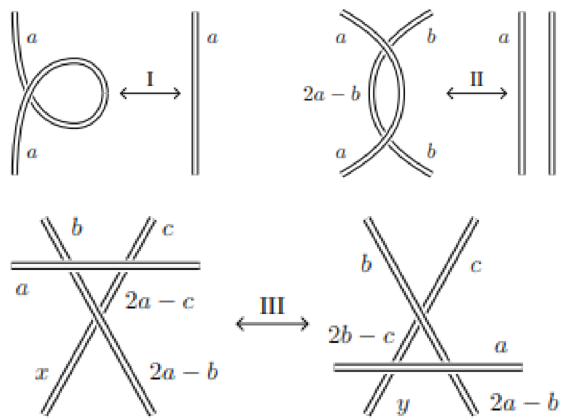

Figure 3. The n-labelings for the Reidemeister moves

The first is trivial, as the n labeling would be $a + a \equiv 2a \pmod{n}$ which is true for all n, and therefore doesn't change the number of n-labelings. The second is also trivial, as the n labeling would be $2a - b + b \equiv 2a \pmod{n}$, which works for all n. For the third, we must simply show that $x \equiv y \pmod{n}$ for all n, which works, as we find that

$$x \equiv 2a - 2b + c \equiv y \pmod{n},$$

which is true for all n. ∎

Thus, we are ready to prove the theorem.

Proof of Theorem 1.3. Note that the trivial 3 component link has no trivial Fox n-labelings. Now we consider the Borromean Rings. We create the following labeling:

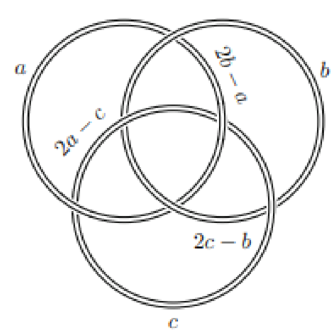

Figure 4. The Labeling of Borromean Rings

To satisfy the crossing relation, we must have

$$2(2b-a) \equiv c + (2a-c) \pmod{n}$$
$$2(2c-b) \equiv a + (2b-a) \pmod{n}$$
$$2(2a-c) \equiv b + (2c-b) \pmod{n}.$$

This implies that $4a \equiv 4b \equiv 4c \pmod{n}$, meaning that the Borromean rings have trivial n-labelings if n is odd and therefore the Borromean rings are not equivalent to the trivial 3 component link. ■

Now that we have shown that the Borromean Rings are not the trivial link, we can show that the Borromean Rings cannot be formed with circles. This is equivalent to the following theorem.

Theorem 1.6. *If a link consists of pairwise unlinked circles, then the link is trivial.*

Proof. By shifting each circle slightly we can assume that the circles lie in distinct planes, no two of the planes are parallel, and none of the planes spanned by one of the circles contains the center of a second circle. We will say that two circles are linked if one of them intersects the interior of the disk spanned by the other exactly once. Let the circles $C, C' \subseteq \mathbb{R}^3$, D, D' be the disks on the interior of the circles, and H, H' be the planes that they are on. Call the intersection of the planes $L := H \cap H'$. We can see that there are two pairs of intersection points that lie on L, and that they alternate on L.

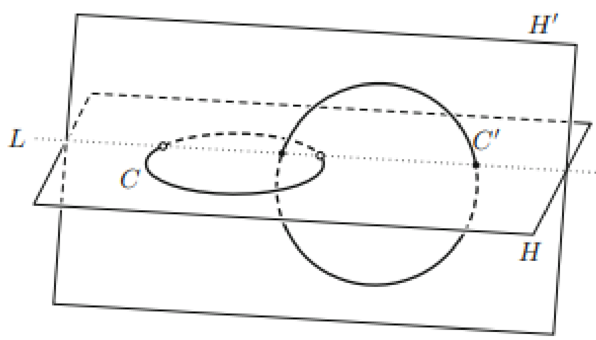

Figure 5. Alternating Intersections

This alternating property characterizes linked circles. To prove the theorem, we take a combination of n circles which are pairwise unlinked and build hemispheres about them. This can be done by taking any point on the disk and raising it by a certain height $h(x)$. We define this more carefully as follows: for a circle (which is a 1-sphere) $C \subseteq \mathbb{R}^3$ with center c and radius r, there is a 2-hemisphere $S \subseteq \mathbb{R}^4$ which can be represented by the graph

$$x, h(x) \in \mathbb{R}^3 \times \mathbb{R} : x \in D,$$

where $h(x)$ represents the height of each point on the disk from the original plane

$$h(x) := \sqrt{r^2 - |x-c|^2}.$$

The projected image of this dome S onto $\mathbb{R}^3$ is the disk D, so we call this dome *orthogonal above* the disk.

Lemma 1.7. *If two circles $C, C' \subseteq \mathbb{R}^3$ are unlinked, then their domes $S, S' \subseteq \mathbb{R}^4$ do not intersect.*

Proof of Lemma 1.7. We will show that if S and S' intersect, then C and C' are linked, which is equivalent. Assume that S intersects S' are some point (a, b). Then we know that $a \in D$, and also that $a \in D'$, meaning that it is also on L. Now the lifting functions h, h' restricted to the line L define perfect half-circles, with their domain being $D \cap L$, and $D' \cap L$, respectively. Because $a \in L$, we see that the half-circles above $D \cap L$ and $D' \cap L$ intersect (they contain the point (a, b)). This means that the endpoints of the half-circles $D \cap L$ and $D' \cap L$ alternate on L, and therefore the circles C and C' (which contain the same endpoints) also have points that alternate on L, which means that the two circles intersect. ■

Now, back to the configuration of n disjoint circles. We know that the circles are in $\mathbb{R}^3$, and that their respective domes are in $\mathbb{R}^3 \times \mathbb{R}$. So we take some $t \in \mathbb{R}$, which represents the extra coordinate that defines the dome. This t can be interpreted as time. We take slices of the dome $\mathbb{R}^3 \times \{t\}$ so as we increase t, we are essentially shrinking the circles while staying on the same domes that were built around the initial circles (which do not intersect due to **Lemma 1.7**). This means that while shrinking the circles, the circles stay disjoint. Note that while shrinking each of these circles, the center of the circle and the plane that contains each of the circles do not change. Then, we can stop the shrinking process when the circles are so small that they do not intersect any of the planes of the other circles, and are therefore completely separate, and thus their link is trivial. ■

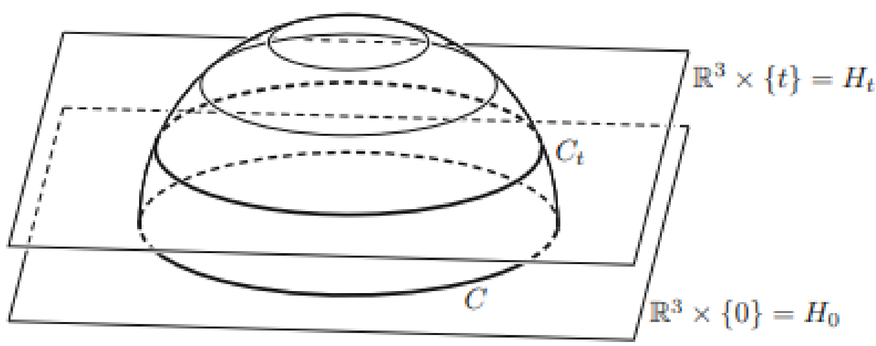

Figure 6. Hemisphere along which the circle is shrunk

Using the combination of the two theorems, we see that the Borromean rings are non-trivial, and that pairwise unlinked circles form only trivial links, and we can therefore conclude that circles cannot form the Borromean rings.

2. Knot and Link complements, and Hyperbolic Geometry

2.1. **Topology of Knot and Link complements.**

Definition 2.1. An $n-$*manifold* is a space that locally resembles n-dimensional Euclidean space. By locally, we mean that if we take a point in the manifold, the space around it is indistinguishable from Euclidean space. For example, a sphere is a 2-manifold, as the space around every point on the sphere locally resembles the Euclidean plane. Some things exist in 3-manifolds, and one convenient 3-manifold is the 3-*sphere* S^3. This is the analogue of a conventional sphere (called the 2-sphere) but one dimension up.

Definition 2.2. A *knot* is a smooth embedding of the circle S^1 in S^3. Similarly, a *link* is a smooth embedding of some number of circles in S^3.

Definition 2.3. The *complement* of a knot is the space in which the knot is removed. With a knot embedded in S^3, this is $S^3 - K$. Notice that the knot complement is also a 3-manifold.

Theorem 2.4. *The* 3*-sphere is* $\mathbb{R}^3$ *with a point at infinity*

Proof. To understand this, we will discuss the lower dimensional version of this.

Lemma 2.5. *The* 2*-sphere is a* $\mathbb{R}^2$ *with a point at infinity*

Proof. We will actually think about this in reverse. If we remove a point from the 2-sphere, or just a conventional sphere, we can easily see that we are left with something homeomorphic to the Euclidean plane. ∎

Applying similar logic, if we remove a point from the 3-sphere, we are left with something that is homeomorphic to Euclidean space, which is $\mathbb{R}^3$. ∎

We call S^3 the *one point compactification* of $\mathbb{R}^3$.

Definition 2.6. A *homeomorphism* is a deformation of a topological spaces. Two topological spaces are said to be *homeomorphic* if one can be deformed into another in any way.

For example, any knot that exists in 3 dimensions is homeomorphic to a circle (or the unknot), as it can be deformed by the shape passing through itself.

Definition 2.7. Two knots are said to be *isotopic* if one can be deformed into another without passing through itself (similarly with any topological space).

Gordon and Luecke proved the following theorem about knot complements:

Theorem 2.8. *If the complement manifold of two knots are homeomorphic, then the knots are equivalent up to isotopy and mirror images. (1989)*

The proof of this theorem is beyond the scope of this paper. However, this does give a useful way of classifying knots based on their complements.

This theorem is not true for links. J.H.C Whitehead proved that there are infinitely many link complements homeomorphic to the Whitehead Link.

2.2. **Hyperbolic Knots and Links.**

Definition 2.9. In *hyperbolic geometry*, all of Euclid's postulates hold with the exception of the parallel postulate. There are infinitely many parallel lines. There are several ways to model this in two dimensions: by straight lines in a disk (the Klein model), arcs of circles perpendicular to the boundary of a disk (the Poincaré model), or semicircles perpendicular to the x-axis in the upper half plane.

We call the hyperbolic plane the *hyperbolic metric* of 2 dimensional Euclidean space. In fact, this can be extended to other 2-manifolds in the same manner. The metric of a manifold only changes how angles and distances are measured, and therefore the properties which define a 2-manifold are unchanged. However, knots live in 3 dimensions, so we must generalize this definition of the hyperbolic metric to 3-manifolds.

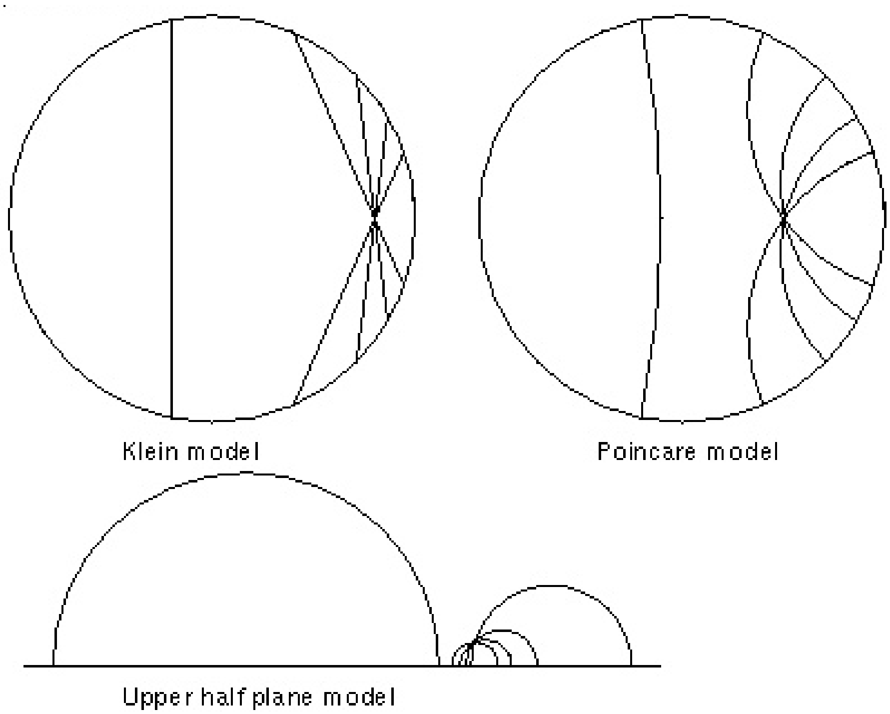

Figure 7. Modeling the hyperbolic plane

One way to visualize this is as follows. Consider a piece of glass lying on a table such that the speed of light n is proportional to the height above the table. The minimal path for the light is a semicircle perpendicular to the plane of the table or a line perpendicular to the plane of the table. This is the basic definition of the upper half space model of hyperbolic space. One of the fundamental discoveries of Bill Thurston is the following.

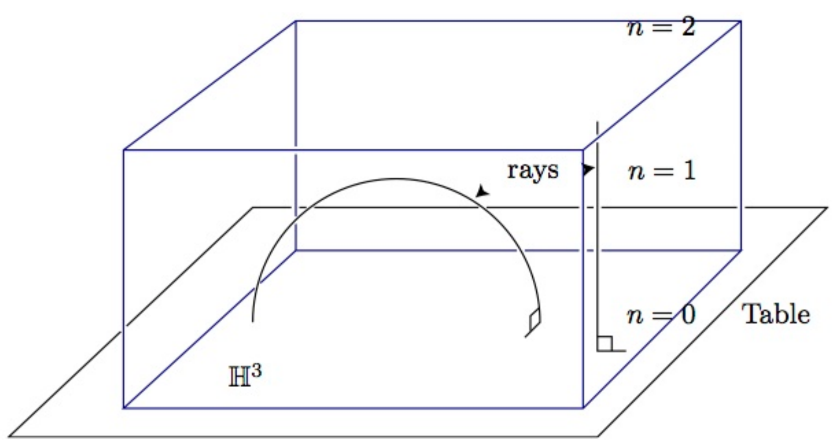

Figure 8. Upper half space model of hyperbolic space

Theorem 2.10. *A knot is a torus knot (sitting on the surface of a torus), a satellite knot (multiple knots tied together), or its complement is hyperbolic.*

The proof of this is outside the realm of this paper. A similar trichotomy applies to links.

Definition 2.11. The *hyperbolic volume* of a knot complement is an important invariant and can be calculated by integrating $1/n^3$ over the region.

As stated earlier, links also have hyperbolic complements. The Whitehead link and the Borromean rings are both have hyperbolic complements. A theorem by George Mostow implies that a hyperbolic link admits a unique hyperbolic structure of finite volume. Thus, the hyperbolic structure becomes an invariant of the link complement.

Definition 2.12. An *ideal* polyhedron in hyperbolic space is the hyperbolic analogue of polyhedron.

The Borromean rings embedded in S^3 have a complement which is two ideal octahedron glued together at a face.

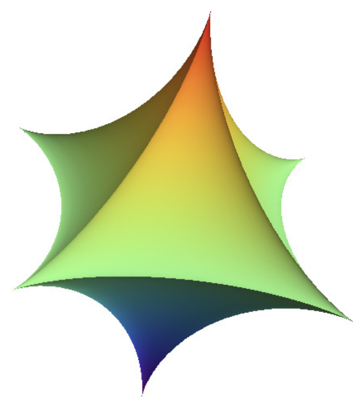

Figure 9. A single ideal octahedron

The volume of the complement of a link is a very powerful invariant. Thurston and Jørgensen proved the following about them:

Theorem 2.13. *There are finitely many link complements of a certain volume.*

We note however, that there may be infinitely many links of homeomorphic complements. There are several other powerful features of link complements, one of which is illustrated below.

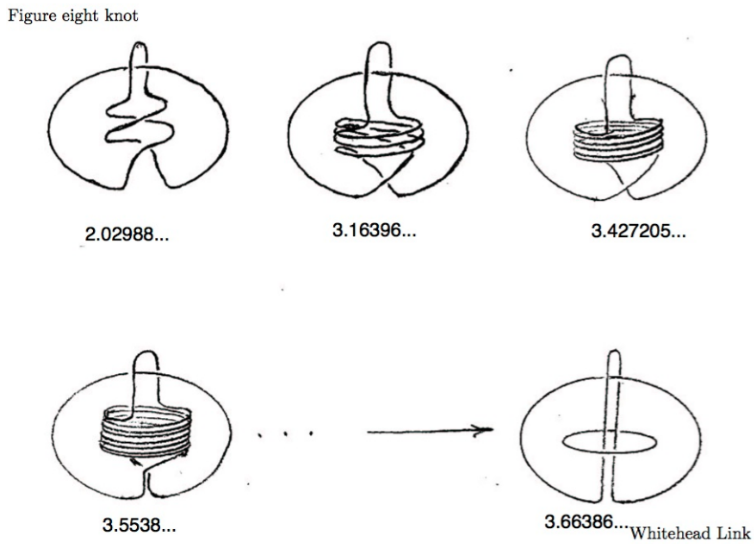

Figure 10. Link complement volumes are "well ordered"

3. Further Research

We showed that the Borromean Rings cannot be formed by 3 perfect circles. However, the following is still open:

Question 3.1. *Is there any other shape that is homeomorphic to the circle, such that three of them cannot form the Borromean rings?*

There are several interesting questions about link complements.

Question 3.2. *The minimal volume for a link comprised of* $1, 2$ *and* 4 *circles is known. What is the minimal volume for a link comprised of* 3 *circles. More in general, what is the minimal volume for a link comprised of* n *circles?*

Volume	Proved by / when	link
$2.029\ldots$	Cao-Meyerhoff 2001	
$2.828\ldots$	Gabai-Meyerhoff-Milley 2009	 and $(-2, 3, 7)$ pretzel
$3.663\ldots$	Agol 2010	
$7.327\ldots$	Yoshida 2012	

Figure 11. The minimal volumes for the links comprised of $1, 2, 4$ circles

4. References

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