The Slope Problem

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Abstract

In a configuration of $n \geq 3$ points where not all points lie on a line, we would like to find the minimum number of slopes when we connect the points. In this paper, we will review the proof in [1] of the fact that, when n is odd, the minimum number of slopes is $n-1$, and when n is even, the minimum number of slopes is n.

1 Introduction

Given $n \geq 3$ points that are not all in a line, what is the minimum number of slopes we can get if we connect these points? Certain configurations of n points can have less slopes than others. In this configuration, where $n = 4$, there are 6 slopes: -3, -$\frac{1}{2}$, 0, $\frac{2}{3}$, 2, 3 (Figure 1).

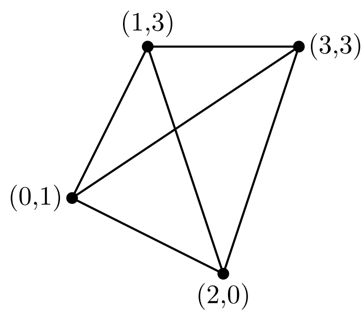

Figure 1: $n = 4$ and $M = 6$.

But, we can find a configuration of $n = 4$ points with fewer slopes. Let M be the minimum number of different slopes. We want to find M for $n = 4$. Three of the points, A, B, and C, form a non-degenerate triangle since not all 4 points lie on a line. This implies that all 3 sides have different slopes, so M has to be at least 3. Let l, m, and n be the lines AB, AC, and BC respectively, extended in both directions. If D lies on l, we can immediately see that $DC \nparallel AC$ and $DC \nparallel BC$. In addition, $DC \nparallel AB$ unless C lies on the line containing A, B, and D which cannot happen. So, DC has a different slope from AB, AC, and BC. If D lies on m, then DB in not parallel to AB, AC, and BC, and if D lies on n, then DA in not parallel to AB, AC, and BC. This implies that if D is a point on l, m, or n, there are at least 4 different slopes.

Suppose there is a position for D such that there are only 3 different slopes. Then, $DA \nparallel AB$ since D does not lie on l, and $DA \nparallel AC$ since D does not lie on m, so DA has to be parallel to BC. Similarly, DC has to be parallel to AB, and DB has to be parallel to AC. So D must be at the intersection point of the line parallel to AB passing through C, the line parallel to AC passing through B, and the line parallel to BC passing through A. But this is not possible since the 3 lines form a non-degenerate triangle (Figure 2). Hence, there must be at least 4 different slopes.

Consider 4 points that form a square when connected. Then there are 4 slopes. When we add a fifth point at the intersection of the diagonals of the square, then there is still 4 slopes (Figure 3).

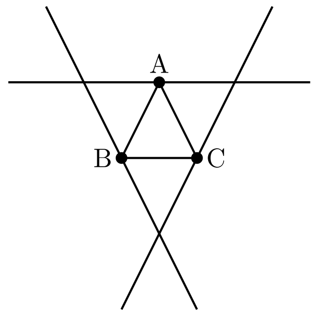

Figure 2: An example of the line parallel to AB passing through C, the line parallel to AC passing through B, and the line parallel to BC passing through A.

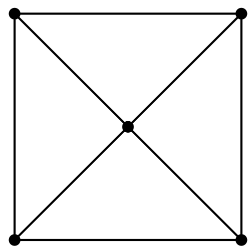

Figure 3: If $n = 5$, this configuration gives the smallest number of slopes.

Theorem 1. *[1] If $n \geq 3$ points in the plane do not lie on one single line, then they determine at least $n-1$ different slopes, where equality is possible only if n is odd and $n \geq 5$.*

2 Sequence of permutations

Proof. We will first show why the minimum number of different slopes is $n-1$ when n is odd. Consider a line segment l that contains $n-2$ points, $P_1, \ldots, P_{n-2}$ that partition l into $n-3$ pieces. We can create a perpendicular bisector m to l with two points, B and C equidistant from l. The point $P_{\frac{n-1}{2}}$ is the point at the intersection of l and m and we can call it A (Figure 4). If we connect $P_1, \ldots, P_{n-2}$ to C, we get the line segments $P_1C, \ldots, P_{n-2}C$, so there are $n-2$ different slopes. The line l has a different slope from all of the $n-2$ slopes since C doesn't lie on l. The other two lines, BC and BA, have the same slope as AC since A lies on BC, and we have already counted AC part of the $n-2$ slopes. We can also see the lines connecting B to $P_1, \ldots, P_{n-2}$ have the same slopes as the lines connecting C to $P_1, \ldots, P_{n-2}$. Therefore, when n is odd, there is a configuration that has $n-1$ different slopes.

If n is even, we can create a similar configuration to the one where n is odd, that has n slopes. Consider a line segment l that contains $n-2$ points, $P_1, \ldots, P_{n-2}$ that partition l into $n-3$ pieces. We can create a perpendicular bisector m to l connecting two points, B and C that are equidistant from l. There are $n-2$ slopes when we connect C to $P_1, \ldots, P_{n-2}$, line l has a different slope, and BC has a different slope. So, there are n slopes.

Now, we need to prove that when n is even, the minimum number of slopes is n. Given a configuration with n points, let t be the number of slopes. Then $t \geq 2$ since not all points can lie on a line. We can begin by drawing a circle around the configuration of points. Let X be a point at the base of the circle. Consider a line tangent to the circle at X. We can assume, without loss of generality that in the configuration, no line connecting two or more points is perpendicular to the tangent. Next, project the points of the configuration

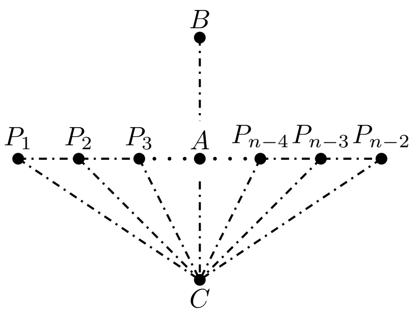

Figure 4: Here, n is odd and $M = n - 1$.

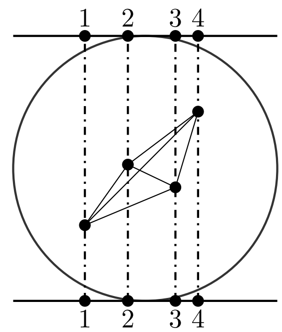

Figure 5: For this configuration of points where $n = 4$, we can project the points onto the tangent, when the tangent is at the base of the circle and the tangent is at the top of the circle.

onto the tangent and label them $1, \ldots, n$ in the order they appear. Call this projection $\sigma_0 = 1 \ldots n$, the first permutation. We can start rotating X counterclockwise around the circle.

Let P and Q be points on the circle such that when we project the points of the configuration onto the tangent at P, we get the permutation, σ_i, and when we project the points of the configuration onto the tangent at Q, we get the permutation, σ_{i+1}. From P to Q, the permutation changes. If the permutation changes, then at least two numbers in the permutation swap, which means that the order of the projections of at least two points in the configuration swap. Let A and B be two of the points in the configuration that are being projected. At some point M in between P and Q, A and B have the same projection when projected on to the tangent at M. We can see that this happens when the tangent is perpendicular to the line connecting A and B.

This implies that the number of times that the permutation changes is the same as the number of times that the tangent is perpendicular to a line connecting at least two points in the configuration. So, the number of permutations in the sequence is equal to the number of slopes in the configuration of points. By rotating X 180° around the circle, we get this sequence of permutations: $\sigma_0 \to \ldots \to \sigma_t$. We notice that the first permutation in the sequence is $\sigma_0 = 123...n$ and the last is $\sigma_t = n...321$, and this is easy to see if we rotate the circle along with the configuration of points (Figure 5).

Also, by the end of the sequence, every pair of numbers i and j where $1 \leq i \leq j \leq n$ are swapped exactly once. This is because, from σ_0 to σ_t, the places where i and j swap are when the tangent is perpendicular to the line connecting i and j, and this happens at both ends of the line. But only one occurs in the 180° section of the circle. Now, if we rotate X 360° around the circle and project the points in the configuration onto the tangent, we get the following sequence:

$$\sigma_0 \to \sigma_1 \to \ldots \to \sigma_{t-1} \to \sigma_t \to \sigma_{t+1} \to \ldots \to \sigma_{2t}$$

We can see that the term σ_{i+t} is the reverse of the term σ_i where $1 \leq i \leq t$ and σ_0 is equal to σ_{2t}.

Definition 1. The *partition* divides each permutation with an even number of elements, into two equal parts, a left side and a right side. If a permutation has n numbers, when we partition the permutation, the left side and right side have $\frac{n}{2}$ numbers.

We can partition each permutation and see how many numbers cross the partition. A move represents the transition from one permutation to the next.

Definition 2. A move between two permutations is a *crossing move* if the numbers that get swapped cross the partition.

Definition 3. If the crossing move is such that exactly d numbers on one side of the permutation cross the partition and at least d numbers on the other side of the permutation cross the partition, then the crossing move has *order* d.

For example, in a configuration, consider the permutation $\sigma_2 = 213564$ where the points 1,3,5 and 6 lie in a line. When we cross the line, we get the permutation $\sigma_3 = 265314$. If we partition the two permutations, we can see which numbers cross the partition.

$$\sigma_2 = 213|564 \to 265|314 = \sigma_3$$

where "|" is the partition. In this crossing move, 3 swaps with 5 and 1 swaps with 6, so the order of this crossing move is $d = 2$. The crossing move from $\sigma_5 = 652|341$ to $\sigma_6 = 654|321$ swaps 2 with 4, so this move has order $d = 1$.

Definition 4. A *touching move* is a move that reverses at least one group of numbers where one of the numbers in each group is adjacent to the partition, and no numbers cross the partition.

For example,

$$\sigma_0 = 123|456 \to 213|546 = \sigma_1$$

Here, 4 swaps with 5, and 4 is adjacent to the partition.

Definition 5. An *ordinary move* is a move that is neither a crossing move nor a touching move. This move swaps numbers that aren't adjacent to the partition and none of the numbers cross the partition.

For example,

$$\sigma_1 = 213|546 \to 213|564 = \sigma_2$$

Here, 4 swaps with 6 and both 4 and 6 are not adjacent to, and do not cross the partition.
So, every move is either a crossing, touching, or ordinary move. Let $C(d)$ denote a crossing move of order d, T denote a touching move, and O denote an ordinary move.

Now, we need to prove three things.

Lemma 2. *Let d and e be orders of crossing moves. Then,*

(*a*) *There is a touching move between one crossing move and the next.*

(*b*) *There is at least d ordinary moves between a crossing move and the next touching move.*

(*c*) *There is at least e ordinary moves between a touching move and the next crossing move.*

Definition 6. A *substring* is a group of numbers that are being swapped when we go from one permutation to the next.

Every move acts on a line of points in a configuration, and at the starting position, the points were projected onto the tangent and labeled in increasing order. For example, in a configuration with n points, let k points all lie on a line l where $k < n$. When the tangent is not perpendicular to l, we can project the points onto the tangent and label them $a_1, \ldots, a_k$ in the order they appear. For this position of the tangent, we have the substring, $a_1 \ldots a_k$. When we cross the point where the tangent is perpendicular to l, the projection of the k points onto the tangent reverses and we get the substring, $a_k \ldots a_1$ (Figure 6).

Before a crossing move of order d, we have a substring whose numbers are in increasing order. After the crossing move acts on that substring, the numbers in the substring become in decreasing order. Since the

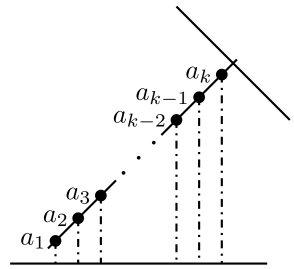

Figure 6: The points $a_1, \ldots, a_k$ lie on a line l. The two lines represent the two positions on the tangent.

substring has numbers on both side of the partition, there is no increasing substring that crosses the partition, so the next move cannot be a crossing move.

Consider the part of the decreasing substring which has d numbers on both sides of the partition. Call the d numbers in the part of the substring on one side of the partition, $a_1, \ldots, a_d$, and the d numbers in the part of the substring on the other side of the partition, $b_d, \ldots, b_1$.

The next touching move will swap a_d with an adjacent number and b_d with an adjacent number, without any number crossing the partition. But, the next move cannot be a touching move since a_d and b_d are in a decreasing substring. The numbers that are possible to swap are a_1 and b_1.

First, an ordinary move will swap a_1 and b_1 with numbers that are adjacent to them that are not one of $a_1, \ldots, a_d$ and $b_d, \ldots, b_1$. Now, there are $d-1$ numbers on both sides of the partition. The next ordinary move will swap a_2 and b_2 with numbers adjacent to them that are not one of $a_1, \ldots, a_d$ and $b_d, \ldots, b_1$. Ordinary moves must take place until we have one number on each side of the partition. When we have one number on each side, the next move can be a touching move. So, there must be at least $d-1$ ordinary moves after a crossing move and before the next touching move.

After a touching move, let the next crossing move have order e. Since the crossing move acts on an increasing substring, we can consider the part of the increasing substring which has e numbers on both sides of the partition. Call the e numbers in the part of the substring on one side of the partition, $c_1, \ldots, c_e$, and the e numbers in the part of the substring on the other side of the partition, $f_e, \ldots, f_1$. The previous move has to have been an ordinary move that swapped c_1 and f_1 with numbers adjacent to them that are not one of $c_1, \ldots, c_e$ and $f_e, \ldots, f_1$. Working backward analogously to what we did for (a), we can see that there must be at least $e-1$ ordinary moves that had taken place before the crossing move.

By the end of the sequence of permutations, all the numbers will have crossed the partition at least once since, when X rotates 180° around the circle, we get the sequence of permutations,

$$\sigma_0 \to \sigma_1 \to \ldots \to \sigma_{t-1} \to \sigma_t$$

so we have crossed all the lines exactly once, which means that every pair of numbers in the permutation have swapped exactly once. Let the orders of the crossing moves be $x_1, \ldots, x_c$. If the crossing move has order x_i, it means that $2x_i$ numbers crossed the partition. Since all n numbers have crossed the partition at least once,

$$\sum_{i=1}^{c} 2x_i \geq n$$

A touching move can only take place between two crossing moves. If there is only one crossing move, and no touching moves, then all the points in the configuration must all lie on a line, but this is not allowed. So, there must be a touching move. Assume, without loss of generality, that X is at a point where the first move is a touching move. The length of the sequence of permutations is equal to the sum of the number of touching, crossing, and ordinary moves. First, there are c crossing moves (C). Next, since there is one T between every pair of C's, and the first move is a touching move, there are at least c T's. Finally, before and after every C, there are $x_i - 1$ O's, so the total number of O's are

$$\sum_{i=1}^{c}(2x_i - 2) = \sum_{i=1}^{c} 2x_i - 2c$$

When we add the total number of C's, O's, and T's, we get

$$t \geq \sum_{i=1}^{c} 2x_i \geq n$$

So, when n is even, the minimum number of slopes is n. ■

References

[1] M. Aigner, G. M. Ziegler, *Proofs from the Book*, Springer-Verlag, Berlin & Heidelberg, 2010.