SOLVING THE DINITZ PROBLEM

ANTHONY LEONG

1. Introduction

The Dinitz Problem is a coloring problem raised by Jeff Dinitz in 1978. 15 years later, Fred Galvin was able to solve this problem using graph theory. A result of the Dinitz problem is that it expanded much of the current understanding of modern graph theory.

Consider n^2 cells arranged in a $n \times n$ square and let (i,j) denote the cell in row i and column j. Suppose that for every cell (i,j) we are given a set C(i,j) of n colors. Is it then always possible to color the whole array by picking for each cell (i,j) a color from its set C (i,j) such that the colors in each row and each column are distinct?

To understand this problem, let's consider a simple case where each cell has the same color set of $\{1, \dots, n\}$. We can use Latin squares to solve this.

1	2	3
2	3	1
3	1	2

Figure 1. Here is an example of a Latin Square with n=3.

In a Latin square, it is always possible to fill up the cells so that no cells in the same row or column have the some color. In Figure 1, we could fill each cell in the 3×3 box with a number 1 to 3. This can be easily shown because when we fill out a row, which is n-cells long, from left to right, each subsequent cell has 1 less choice than the one before. Once we get to the last cell, there will be exactly one choice left, thus, we have constructed a row which satisfies the criteria for the Dinitz Problem. We can construct the next row the same way, except, we cannot repeat the same color as the cell directly above a cell in the second row. To do this, we can use the same pattern as the first row except that each cell is shifted over one cell to the right, and the rightmost cell will be moved below the leftmost cell in the first row. We can continue this for n-1 steps after the first row which will be just enough to construct an $n \times n$ square where no cell has the same color any others in the same row or column. This is done in Figure 1, except that the cells are moved one cell to the left.

Notice that it is impossible to color the cells with an amount of colors less than n. We would not even have enough colors to construct a row that satisfies this criteria.

The Dinitz problem is harder that filling the cells as Latin squares because each cell has a designated set, C(i,j), of n colors. The set $\bigcup_{i,j} C(i,j)$ may contain more than n colors. For example, let's try n=2. In the picture below, it is not always possible to pick a coloring. If we pick 2 for the second cell in the first row, it will be impossible to create a coloring.

Date: August 17, 2020.

$\{1,2\}$	$\{2,3\}$
$\{1,3\}$	$\{2,3\}$

To solve the Dinitz Problem, a little bit of background knowledge on graph theory is necessary. Let us construct the graph by assigning a vertex to each cell. We will use edges to connect the vertices to other vertices in the same row and column. The notation we will use to denote the graph will be G=(V,E) where $|V|=$ number of vertices, $|E|=$ number of edges. $\chi(S)$ is the chromatic number of the graph (the smallest number of colors that one can assign to the vertices such that adjacent vertices-vertices connected by an edge- will receive different colors). By coloring the vertices, we are partitioning the vertices into their own groups or classes. A set $A \subseteq V$ is independent if there are no edges between the vertices within the set A.

2. List Colorings

An issue we ran into when comparing the cases between the Dinitz problem and the basic case with each cell having the same n colors is that some cells have a different set of n-colors.

Definition 2.1. A *list coloring is a coloring* $c : V \to \bigcup_{v \in V} C(V)$ where $c(v) \in C(v)$ for each $v \in V$.

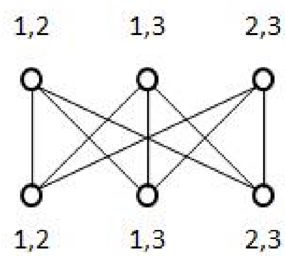

Figure 2. List coloring of a complete bipartite graph $K_{3,3}$.

The function $C : V$ assigns a designated set of colors to each $v \in V$. This way, each vertex will have a list of colors as in the original question. Figure 2 is a list coloring because each vertex is assigned a set of colors, C(v).

Definition 2.2. *List chromatic numbers* $(\chi_\iota(G))$ is the smallest number k such that, for any list of color sets C(v) with $|C(v)| = k$ for all $v \in V$ there always exists a list coloring.

Notice that if we pick one color in Figure 2, there cannot be a coloring since there are only 3 colors, and each vertex is connected to 3 colors. That means that the list chromatic number is greater than 3.

An interesting concept that could also be applied to list colorings is choosability in graphs. We can say that a graph is k-choosable if a coloring exists for all vertices if the number of elements in each vertex is $\geq k$.

$(\chi_\iota(G)) \leq |V|$ since having the same number of colors in each cell as the number of vertices would allow each vertex to have a color different from every other vertex in the graph.

$\chi(G) \leq \chi_\iota(G)$ since ordinary coloring of the graph is a more optimal method of coloring the graph than list coloring. This is because ordinary coloring is list coloring when each cell has the same colors.

Let (S_n) denote the graph corresponding to the $n \times n$ grid. Since we know that the coloring of the $n \times n$ grid must have at least n-colors, we can restate the Dinitz problem:

Proposition 2.3. *Is it true that* $\chi_\iota(Sn) = n$?

Let's consider the graph $G = K_{4,2}$.

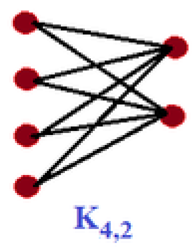

Figure 3. Complete bipartite graph pairing 4 vertices on one side with two vertices on the other

Each vertex has a chromatic number of 2 since we would only need one color for the two right vertices and the other color for the four left ones. In Figure 3, let us assign color sets to the left vertices from top to bottom as $\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}$ and the right ones from top to bottom as $\{1,2\}, \{3,4\}$. This way, we can assign the right vertices as $1|3$, $1|4$, $2|3$, and $2|4$. Each of these sets can appear as a color set on the left side, so we would need at least a list chromatic of 3 or more.

Now, let's get back to solving the Dinitz Problem. An important step in finding a solution was made by Jeanette Janssen in 1992. It was proven that $\chi_\iota(S_n) \leq n + 1$. This was combined with two lemmas by Fred Galvin to finally solve the Dinitz Problem.

3. Directed Graphs and Kernels

Definition 3.1. The *degree of a vertex*, v, denoted as d(v), is the number of edges connected to that vertex.

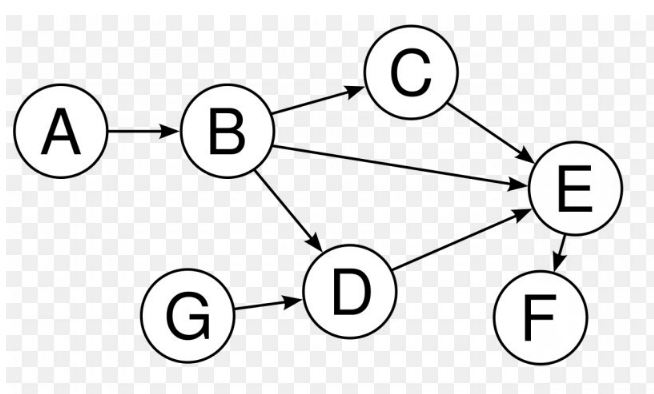

Figure 4. A directed graph with vertices labeled A through G

In S_n, the degree of each vertex is 2n-2 since each vertex is connected to the n-1 other vertices in its row and n-1 other vertices in its column.

Definition 3.2. For a subset $A \subseteq V$, we denote the *subgraph* by G_A which has A as vertex set and which contains all edges of G between vertices of A.

Definition 3.3. H is an *induced subgraph* of G if $H = G_A$ for some A.

Definition 3.4. *Directed graphs* ($\vec{G} = (V, E)$) are graphs where every edge e has an orientation.

Figure 4 is an example of a directed graph, and the direction of the arrows indicate the orientation.

Definition 3.5. *e=(u,v)* means that there is an arc e, also denoted by $u \to v$ whose initial vertex is u and whose terminal vertex is v.

In figure 4, at the left, the edge between A and B has A as the initial vertex and B as the terminal vertex.

Definition 3.6. The *outdegree* ($d^+(v)$) counts number of edges with v as initial vertex.

If we use vertex E, the outdegree is only 1.

Definition 3.7. The *indegree* ($d-(v)$) counts number of edges with c as a terminal vertex

If we also use vertex E as an example, the indegree is 3.
An important observation to notice is that: $d+(v)+d-(v) = d(v)$

Definition 3.8. Let $\vec{G} = (V, E)$ be a directed graph. A *kernel* $K \subseteq V$ is a subset of the vertices such that:
(i.) K is independent in G, and
(ii.) for every $U \notin K$, there exists a vertex $v \in K$ with an edge $u \to v$.

In Figure 4, the set $\{A, C, D, F\}$ is a kernel.

Lemma 3.9. *Let $\vec{G} = (V, E)$ be a directed graph, and suppose that for each vertex $v \in V$, we have a color set $C(v)$ that is larger than the outdegree, $|C(v)| \geq d+(v)+1$. If every induced subgraph of $\vec{G}$ possesses a kernel, then there exists a list coloring of G with a color from $C(v)$ for each v.*

Proof. We will use induction on $|V|$. For $|V| = 1$ we are done.
Choose a color $c \in C = \bigcup_{v \in V} C(v)$ and set $A(c) = \{v \in V : c \in C(v)\}$.
By hypothesis, the induced subgraph $G_{A(c)}$ possesses a kernel K(c). Now we color all $v \in K(c)$ with the color c since K(c) is independent, and delete $K(c)$ from G and c from C. Let G^* be the induced subgraph of G on $V \setminus K(c)$ with $C^*(v) = C(v) \setminus c$ as the new list of color sets. Notice that for each $v \in A(c) \setminus K(c)$, the outdegree $d^+(v)$ is decreased by at least 1 (due to condition (ii) of a kernel). So $d^+(v) + 1 \leq |C^*(v)|$ still holds in $\vec{G^*}$. The same condition also holds for the vertices outside $A(c)$, since in this case the color sets $C(v)$ remain unchanged. The new graph G^* contains fewer vertices than G, and we are done by induction. ∎

Now, we have to find an orientation of the graph S_n with outdegrees $d+(v) \leq n-1$ for all v and which ensures the existence of a kernel for all induced subgraphs.

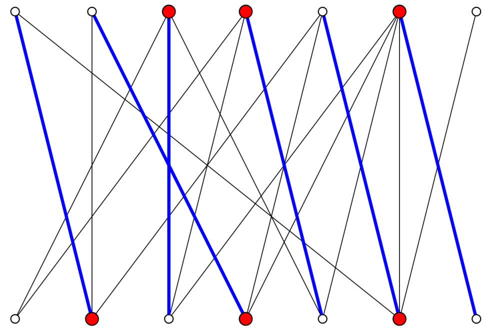

Figure 5. An example of a bipartite graph where the blue lines are matchings, and the uncolored lines are potential matchings (N(v) for each vertex, v

4. Stable Matchings

Definition 4.1. A *bipartite graph* $G = (X \cup Y, E)$ is a graph with the following property: the vertex set V is split into two parts X and Y such that every edge has one endvertex in X and the other in Y. (Can be colored with two colors (X and Y))

Figure 5 is an example of a bipartite graph, and the vertices are split into the top and bottom sets.

Definition 4.2. A *matching* M in a bipartite graph $G = (X \cup Y, E)$ is a set of edges such that no two edges in M have a common endvertex.

An analogy to best explain the concept of stable matchings will be marriage. Each vertice represents a person. Let X be a set of men and Y be a set of women. $uv \in E$ means that u and v could marry, and bigamy is not allowed. In real life, a person may have preferences to whom he or she would want to marry. This gives a ranking system as follows: In $G = (X \cup Y, E)$, we assume that for every $v \in X \cup Y$ there is a ranking of the set N(v) of vertices adjacent to v, $N(v) = (z_1) > (z_2) > \ldots > (z_d(v))$.Thus, z_1 is the top choice for v followed by z_2, and so on.

Definition 4.3. A matching M of $G = (X \cup Y), E$ is called *stable* if the following condition holds: Whenever $uv \in E \setminus M$, $u \in X$, $x \in Y$, then either $uy \in M$ with $y > v$ in N(u) or $xv \in M$ with $x > u$ in N(v), or both.

In the analogy, a matching would not be stable if there is a preference where both vertices would prefer each other over their current partner if married.

Lemma 4.4. *A stable matching always exists.*

Proof. Let's create an algorithm. In the first stage all men $u \in X$ propose to their top choice. If a girl receives more than one proposal she picks the one she likes best and keeps him on a string, and if she receives just one proposal she keeps that one on a string. The remaining men are rejected and form the reservoir R. Next, all men in R propose to their next choice. The women compare the proposals (together with the one on the string, if there is one), pick their favorite and put him on the string. The rest is rejected and forms the new set R. Now the men in R propose to their next choice, and so on. A man who has proposed to his last choice and is again rejected drops out from further consideration (as well as from the reservoir). Clearly, after some time the reservoir $R = \emptyset$, which ends the algorithm. This is suffice to form a stable matching.

Notice first that the men on the string of a particular girl move there in increasing preference (of the girl) since at each stage the girl compares the new proposals with the present mate and then picks the new favorite. Hence if $uv \in E$ but $uv \notin M$, then either u never proposed to v in which case he found a better mate before he even got around to v, implying $uy \notin M$ with $y > v$ in N(u), or u proposed to v but was rejected, implying $xv \in M$ with $x > u$ in N(v). But this is exactly the condition of a stable matching. ■

5. Galvin's Solution to the Dinitz Problem

Theorem 5.1. *We have $\chi_\iota(S_n) = n$ for all n.*

Proof. As before we denote the vertices of S_n by $(i, j), 1 \leq i, j \leq n$. Thus (i, j) and (r, s) are adjacent if and only if $i = r$ or $j = s$. Take any Latin square L with letters from $\{1, 2, ..., n\}$ and denote by L(i, j) the entry in cell (i, j). Next make S_n into a directed graph $\vec{S_n}$ by orienting the horizontal edges $(i, j) \to (i, j^*)$ if $L(i, j) < L(i, j^*)$ and the vertical edges $(i, j) \to (i^*, j)$ if $L(i, j) > L(i^*, j)$. Thus, horizontally we orient from the smaller to the larger element, and vertically the other way around. Figure 6 shows what the directed graph will look like for n=3.

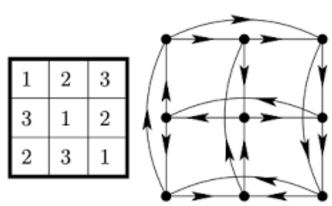

Figure 6. An example of the constructed directed graph for n=3

Notice that we obtain $d^+(i, j) = n - 1$ for all (i, j). This is because each vertex has an outdegree if a number greater than it is in the same row or if a number less than it is in the same column. In fact, if $L(i, j) = k$, then n-k cells in row i contain an entry larger than k, and (k-1) cells in column j have an entry smaller than k.

By lemma 3.9, it remains to show that every induced subgraph of $\vec{S_n}$ possesses a kernel. Consider a subset $A \subseteq V$, and let X be the set of rows of L, and Y the set of its columns. Associate to A the bipartite graph $G = (X \cup Y, A)$, where every $(i, j) \in A$ is represented by the edge ij with $i \in X$,$j \in Y$. In Figure 7, the cells of A are shaded.

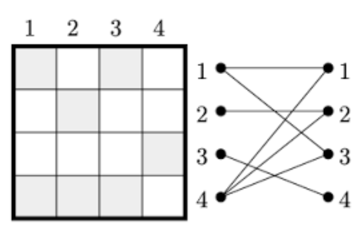

Figure 7. This is a matching with that corresponds to S_n

The orientation on S_n naturally induces a ranking on the neighborhoods in $G = (X \cup Y, A)$ by setting $j^* > j$ in $N(i)$ if $(i, j) \to (i, j^*)$ in $\vec{S_n}$ respectively $i^* > i$ in $N(j)$ if $(i, j) \to (i^*, j)$. By lemma 4.4, $G = (X \cup Y, A)$ possesses a stable matching M. Note first that M is independent in A since as edges in $G = (X \cup Y, A)$ they do not share an endvertex i or j. Secondly, if $(i, j) \in A \notin M$, then by the definition of a stable matching there either exists $(i, j^*) \in M$ with $j^* > j$ or $(i^*, j) \in M$ with $i^* > i$, which for $\vec{S_n}$ means $(i, j) \to (i, j^*) \in M$ or $(i, j) \to (i^*, j) \in M$. The key to what we did is connect the notion of stable matching with the desired subset, A, which matches lemma 3.9. By lemma 3.9, we would have a list coloring. The proof is complete. ■

6. Line Graphs

Let the completed bipartite graph be denoted by $K_{n,n}$ with $|X| = |Y| = n$. Let's allow each edge to be a vertex and connect only the 'edges" that share a common endpoint in $K_{n,n}$. We actually end up obtaining S_n as the new graph since 2n-2 edges also share a common endpoint with any arbitrary edge. We call S_n a line graph of $K_{n,n}$.

Let's denote the line graph of a graph, G, by L(G). We will call H a line graph if $H = L(G)$. (Here, I would like to draw some examples of line graphs)

Does $\chi_\iota(H) = \chi(H)$ hold for every line graph H?

This question has not been answered yet, but with time, we may see more and more progress to figuring it out.

References

[1] Jeannette Janssen, LaTeX*The Dinitz Problem Solved for Rectangles* Bulletin (New Series) of the American Mathematical Society, Volume 29, Number 2, October 1993, ULR: https://www.ams.org/journals/bull/1993-29-02/S0273-0979-1993-00430-0/S0273-0979-1993-00430-0.pdf, Pages 243-249.

[2] Martin Aigner, Gunter Ziegler, LaTeX*Proofs from The Book* Sixth Edition, March 1998, Pages 271-276

[3] Paul Erdos, LaTeX*Choosability in Graphs* ULR: https://web.archive.org/web/20160309235325/http://www.math-inst.hu/ perdos/1980-07.pdf, Pages 126-129