The Friendship Theorem

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1 Introduction

The friendship theorem answers a classic problem of mathematics, specifically graph theory. However, the origin of the question is yet to be discovered. The theorem can be stated as follows:

Theorem 1. *Suppose in a group of people we have the situation that any pair of persons have precisely one common friends. Then there is always a person (the"politician") that everyone is friends with.*

However, to prove this theorem, it will be more helpful to rephrase the theorem using graph theoretic terms. In order to do this, it is important to define what a friendship graph is.

Definition 1.1. A friendship graph is one in which each pair of vertices has precisely one common neighbor.

The theorem can now be rephrased to include a graph theoretic description.

Theorem 2. *Every friendship graph is a windmill graph, or in simpler terms, there is one vertex that is adjacent to all other vertices.*

An example of a windmill graph can be seen in figure 1.

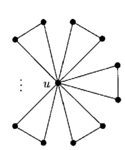

Figure 1: windmill graph

This paper will explore the original proof of this theorem, as well as a few generalizations and a set theoretic application that depends on more relaxed constraints. To do this we study the theorem in terms of more set theoretic concepts. A translation of the theorem in to set theory is as follows:

Definition 1.2. A friendship set is a finite set with a symmetric non- reflexive binary relation, called "on," satisfying: (i) if a is not on b, then there exists a unique element which is on both a and b; (ii) if a is on b, then there exists at most one element which is on both a and b. The relation ship "on to" will be referred to as θ and the relation ship " not on to " will be referred to as θ'

2 The Original Proof

The first proof of the friendship theorem was provided by Paul Erdös, Alfred Rényi and Vera Sós. This proof, is to this day the most accomplished proof provided for this theorem. Before diving into the proof however, it is first necessary that we prove an important condition:

Lemma 1. *A friendship graph G has a C_4 property, which means that there exist no cycles on 4 nodes. Additionally, the distance between any 2 nodes in G is at most 2.*

Proof. Suppose we assume the contrary, and have C_4 as a subgraph of G. Then, there would be 2 vertices u and v that have at least two common neighbors, as two opposite vertices of a square. However, this poses a contradiction to the friendship condition. Additionally, the distance between any two nodes be at most two, since if the distance was any greater, the two vertices have no common friends, which is once a gain a contradiction of the friendship theorem. □

At this point, we can begin the proof of the actual friendship theorem.

Proof. Suppose we assume the contrary, and provide the graph G as a counterexample, meaning there exists no vertex in G that is adjacent to all other vertices. There are two necessary steps for deriving a contradiction, the first involving combinatorics and the second involving some standard results of linear algebra.

1. For the first step, we want to show that $d(u) = d(v)$ for any $u, v \in V$. We can first start by showing that $d(u) = d(v)$ when u and v are nonadjacent vertices. Let $d(u) = k$, where $a_1, a_2, \ldots, a_k$ are the neighbor of u. Of all $a_1, a_2, \ldots a_k$, precisely one a_1, say without loss of generality a_2, is adjacent to v, and a_2 is adjacent to exactly one other a_1, say a_1. The vertex v has the common neighbor a_2 with a_1, and with any $a_i, (i \geq 2)$, a common neighbor $z_i (i \geq 2)$. However, in order for the graph to adhere to the C_4 condition each z_i must be distinct. Hence, we can conclude that $d(v) \geq k = d(u)$, and by symmetry that

$$d(u) = d(v) = k.$$

Now, if we want to count the number of vertices in G we can start by summing over the degrees of the k neighbors of u, which gives us k^2. Since each vertex has one common neighbor with u, we have counted every vertex once except u, meaning the number of vertices in G is

$$n = k^2 - k + 1.$$

2. As stated above, the second step is an application of some standard results in linear algebra. Before we can begin, it is important to note that our k value must be greater than 2, because when $k \leq 2$, we get trivial windmill graphs. Now, suppose we look at the adjacency matrix $A = (a_{ij})$. From step (1) we can see that each row must have exactly k 1's, and since each pair of vertices has exactly one common neighbor, for any two columns, there is exactly one column where both have a 1. Therefore, we get :

$$A^2 = \begin{pmatrix} k & 1 & \ldots & 1 \\ 1 & k & & 1 \\ \vdots & \ddots & \vdots & \\ 1 & \ldots & 1 & k \end{pmatrix} = (k-1)I + J,$$

where I is the identity matrix and J is a matrix consisting purely of 1's. From our above equation, we can then notice that the eigenvalues of A^2 are $k - 1 + n = k^2$ and $k - 1$, which makes the eigenvalues of A, k and $\pm\sqrt{k-1}$. Suppose now, that r of the eigenvalues are $\sqrt{k-1}$ and s of them are $-\sqrt{k-1}$, so that $r + s = n - 1$ where $r \neq s$. Since the sum of the eigenvalues of a matrix are equal to its trace, we can then see that

$$k + r\sqrt{k-1} - s\sqrt{k-1} = 0 \implies \sqrt{k-1} = \frac{k}{s-r}.$$

We now use the fact that if the square root of a natural number is rational, then it an integer (this was proven by Dedekind in 1858). Suppose we let $h = \sqrt{k-1} \in \mathbb{N}$, then

$$h(s-r) = k = h^2 + 1$$

Because h is a divisor of $h^2 + 1$ and h^2, h must be 1, making $k = 1$. However, we already excluded $k = 2$, which mean we have derived a contradiction.

$\square$

3 Significant Applications of the Theorem

A few significant applications of the friendship theorem exists in the realm of combinatorics. For example, in the paper Intersection Properties of Finite Sets, published by H. J. Ryser in 1973 in the Journal of Combinatorieal Theorey, the theorem is used in classfying l-complete classes.

Additionally, when studying friendship graphs in terms of labeling and coloring, there exist many provable properties, such as the fact that friendship graphs have the chromatic number 3, and that the friendship graph F_n is edge-graceful if and only if n is odd.

4 Generalizations of the Friendship Theorem

Now that we have a solid proof of the theorem, we can examine some of the consequences and generalizations of the friendship theorem. First we will generalize this problem to be variable on the number of paths between vertices and the lengths of said paths. We call graphs with path length k and number of paths between nodes l-regularly k-connected graphs, or $P_l(k)$ graphs.

Theorem 3 (Kotzig's Conjecture)**.** *This examines, what would happen if we were to alter the lengths of the paths, k, but keep $l = 1$, so that there is only one path of length k between each every pair of nodes.*

Kotzig was able to prove in 1974, that there exists no $P_1(k)$ graphs where $k > 2$ for $k \leq 8$, and more cases have been proven through $k \leq 33$, but a general proof for all cases is yet to be generated. Now that we have generalize on k, we can also generalize on l.

Theorem 4 (l friendship graphs)**.** *Keeping a path length of 2, we can focus of l-friendship graphs, or $P_l(2)$-graphs which satisfy that every pair of nodes has exactly l common neighbors. We want to now show that Every l-friendship graph G is a regular graph for $l \geq 2$*

Proof. Consider the graph G with its set of vertices V. Now we can take a node v that had degree d, and call L the subgraph of G containing the neighbors of v and L' to be the subgraph containing $V \backslash L$. It then follows that every node in L' has a distance 3 from v. Considering a second node $a \in L$, a has l neighbors in common with v, and therefore has l neighbors in L. Now, suppose $L' = \emptyset$, then a has l neighbors in L as well as v as a neighbor, making $\deg(a) = l + 1$; this also holds true for all of a's neighbors. We know that $d = \deg(v) \geq l + 1$, since $l + 1$ accounts for a and all of its neighbors. However, if $d > l + 1$, then that would mean that L must have some node b that has no neighbors in commons with $a, a_1, a_2, \ldots, a_l$. This would mean any pair a, b has only one node in common v, which violates the l-friendship condition described earlier. Hence, b cannot exist, making all nodes in L have $l + 1$. Since $V = L \cup v$, all nodes in V must also have a degree of $l + 1$ making G a regular graph.

Now, we must consider the case where $L' \neq \emptyset$. In this case, every $x \in L'$ has l neighbors in L, so that v and x have l common neighbors. For some $a \in L$, there are $(d - 1)$ nodes denoted b in L that are not a, making pairs a, b have l common neighbors. There must be $(d - 1)l$ paths from a to any extreme node $b \in L$, of which $(d - 1)$ must go through v and $l(l - 1)$ paths that go through nodes in L. From this, we can see that $(d - l - 1)(l - 1)$ paths go through some node in L'. We now examine $c \in L'$ that is a neighbor of a, is an intermediate node in $l - 1$ paths of length 2 from a to b, since x has $l - 1$ neighbors in L that are not a. From this, it follows that the number of neighbors a has in L' is:

$$\frac{(d - l - 1)(l - 1)}{(l - 1)} = (d - l - 1).$$

We can also deduce that any node in L has $1 + l + (d - l - 1) = d$ neighbors. Now, we want to prove that the same applies for any vertex in L'. If we select any node as v and construct L and L' with the same properties as the earlier ones, we get the following:

$$|L'| = \frac{d(d-l-1)}{l}$$

from the fact that $|L| = d$ and every node in L has $(d - l - 1)$ neighbors in L'. It then follows that

$$|V| = |L| + |L'| + 1 = \frac{d(d-1)}{l} + 1.$$

Since this holds true for any $v \in V$ of degree d, G must be a d-regular graph. $\square$

5 Set Theoretic Application

Earlier in the paper, we introduced a set theoretic phrasing of the friendship theorem (Theorem 3). This statement provides a translation in to sets with a "on to" relationship. The relationship "on to" will be referred to as θ and the relationship " not on to " will be referred to as θ' One major difference in discussing the friendship theorem in terms of sets, is the relaxation of constraints such that it is not necessary for two friends to have a common friend. We will see that there do in fact exist nontrivial sets, that is is sets that do not have a "politician". However the conditions for such sets to exist are rather restrictive.
Provided the set A with a symmetric relation θ, it is considered a friendship set if if adheres to the following conditions:

(i) $a\,\theta'\,a$ for any element a of A

(ii) if $a\,\theta'\,b$, then there exists a unique element $c \in A$, such that $a\,\theta\,c$ and $b\,\theta\,c$

(iii) if $a\,\theta\,b$, there exists maximum on element $c \in A$ such that $a\,\theta c$ and $b\,\theta\,c/$

Additionally, an element $a \in A$ is considered a P- *element* if $a\,\theta\,x$ for all $x \neq a$ in A. A set A is nontrivial if it has more than three elements and there exist to P-elements in A (the equivalent of not having a "politician"). An example of a nontrivial friendship set, is the Peterson Graph(figure 2).

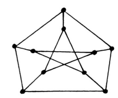

Figure 2: Peterson Graph

Now that we have a clear definition of friendship sets and their conditions, we can start looking at some of their properties.

Lemma 2. *Every element of a nontrivial friendship set is on at least two distinct elements.*

Proof. Suppose we assume the contrary. By definition each element is on at least one element, so we assume that every element is on exactly one element. If and element a is on another element b, then any other elements x is not on a, but is on a common element with a, which in this case is b. However, this would make b a P-element which is not allowed in nontrivial sets, and therefore providing a contradiction. □

Lemma 3. *Suppose a and b are elements of a nontrivial friendship set A with $a \,\theta\, b$. Then, there exists an element $x \in A$ such that $x \,\theta'\, a$ and $x \,\theta'\, b$.*

Proof. Once again, suppose we assume the contrary. Then every element $x \in A$ must be either on a or on b. Because A has no p-elements, there must exist $c, d \in A$ such that $c \,\theta'\, a$ and $d \,\theta'\, b$, which means $c \,\theta\, b$ and $d \,\theta\, a$. Now, $c \,\theta'\, d$ since otherwise a and c would be on the two common elements, specifically b and d. Therefore, there exists an element e that has the following relationships: $e \,\theta\, c$ and $e \,\theta\, d$. This would mean $e \neq a, b$ and e is on either a or b. Now, if e is on a, then a and c are on both b and e. Likewise, if e is on b, then b and d are on both a and e. However, since both of these cases are impossible, we have found a contradiction, hence proving the lemma. □

In addition to these two lemmas, Skala also discusses a few more properties of nontrivial friendship sets whose proof depend on complex linear algebra concepts.

6 Summary

Through this paper, we have examined a classic theorem in combinatorics through its proof as well as few applications and generalizations of the problem. However, many more proofs and applications of the theorem exist. This also includes a variety of open problems, such as the generalization of $P_l(k)$-graphs as a class with variation on both l and k.

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