THE ART GALLERY PROBLEM

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1. Introduction

Let's say you wanted a job at the local museum as a security guard. However, the people at the museum think there are already enough guards. Your job is to convince them that the number of guards they already have is not enough to keep the museum safe at all times.

Definition 1.1. A *museum* is a non-intersecting polygon (not necessarily convex) in a plane.

The museum needs guards to protect its possessions but is unaware how to station the guards so that every location may be observed. A guard is able to keep safe every point that ey can observe by rotating their body but not walking around. In other words, if a guard is stationed at a point G, they are able to guard all points P such that $\overline{GP}$ is entirely contained within the museum.

Example. In **Figure 1**, the museum guard is located at the red point, and the green region is everywhere that is accessible and hence "safe" thanks to the guard.

Question 1.2 (Art Gallery Problem)**.** *What is the minimum number of guards needed to fully protect a museum with n walls?*

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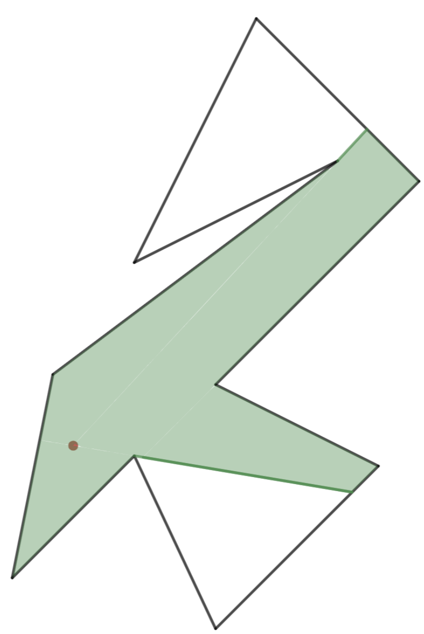

Figure 1. Visibility of a Museum Guard

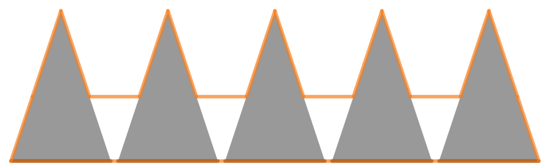

Figure 2. When there needs to be $\frac{n}{3}$ guards

Since the museum wants to save money, they will hire the minimum number of guards possible. How should you convince them that they don't already have enough?

In this paper, we will look at different "museums" to see if there is a discrete formula for the minimum number of guards necessary for a specific shape. This happens to rely on the following theorem.

Theorem 1.3. *The number of guards required to keep safe a museum with n walls (and hence n vertices) safe is at most $\left\lfloor \frac{n}{3} \right\rfloor$.*

We prove this theorem by first showing there always exists a museum with $3n$ walls that requires no less than n guards. We will then show how $\left\lfloor \frac{n}{3} \right\rfloor$ is always sufficient using a triangulating and coloring argument related to graph theory. It will be seen that this certainly is a Book proof, and in fact can already be found in [AZHE10].

Remark 1.4. In the proof of **Theorem 1.3**, we will be using the term 'vertex' in both a geometric and graph theoretical setting. In order to prevent confusion, the 'vertex' is referred to as 'node' when talking about graph theory, and is referred to as 'vertex' only when talking about geometry.

After proving the main theorem, we will explore some variations of the problem such as when all the walls are perpendicular to each other or when we are dealing with more than 2 dimensions.

2. Proof of Theorem 1.3.

Let us first examine why any number less than $\left\lfloor \frac{n}{3} \right\rfloor$ may not be enough guards for a museum with n walls.

In **Figure 2**, we have a "sharp-tooth" patterned museum bordered by the orange walls. Note how each gray triangle must employ a guard within itself in order to be fully secure. Since none of the gray triangles overlap, each must require a distinct guard. In this particular figure, there are 15 walls and hence $\frac{15}{3} = 5$ triangular regions, requiring a minimum of 5 guards. By adding or subtracting one "sharp-tooth" at a time, we can see that for all $n = 3k$ there must be at least $\frac{n}{3} = k$ guards to keep the museum completely safe.

In the cases of $n = 3k + 1$ or $n = 3k + 2$, adding 1 or 2 small walls to any of the corners colored gray will keep the number of guards the same at $\left\lfloor \frac{n}{3} \right\rfloor = k$, but will increase the number of walls accordingly.

We will now see how to guard every museum with n walls with at most $\left\lfloor \frac{n}{3} \right\rfloor$ guards. To do so, we need to first prove the following theorem.

Theorem 2.1. *Every polygon with n sides (concave or convex) can be divided into $n-2$ non-overlapping triangles, or* triangulated*.*

Proof. We first show that a diagonal which divides the polygon into two smaller polygons with $n-r+1$ and $r+1$ sides can always be found. While the existence of diagonals seems obvious, we will prove that there always exists a diagonal that is entirely contained within the polygon.

Note that a polygon with n sides must have a total interior angle sum of $180(n-2)^\circ$, implying that there must be at least 1, and in fact at least 3 interior angles with measure less than 180°. Take a single acute angle from the polygon and name the vertex at the angle's center Q. Next, name the vertices adjacent two Q as P and R. We now end up with one of two different outcomes, which we deal with one at a time.

(1) *No additional vertices of the original polygon are in the region PQR.*

In this case, we simply partition off triangle $\triangle PQR$ from the original polygon, with segment PR being our diagonal. An example is shown in **Figure 3** left.

(2) *Additional vertices are located inside the region PQR.*

In this case, we need to be a little more careful about dividing our polygon. Rather than partitioning triangle $\triangle PQR$ as in case 1, we can instead take a copy of line $\overleftrightarrow{PR}$. Calling this copied line ℓ, we drag it slowly towards vertex Q, such that ℓ is always parallel to the original line $\overleftrightarrow{PR}$. Once line $\overleftrightarrow{PR}$ contains the last vertex of the polygon that is not Q, which we shall call E, we know for sure that no more vertices are in the region bounded by lines ℓ, $\overleftrightarrow{PQ}$, and $\overleftrightarrow{QR}$. *Note that if there are two vertices that are both on line ℓ at this location, just pick one of them to be E.*

Taking the line segment $\overline{QE}$ as the diagonal, we know for sure that it is entirely contained within the polygon. An example of this scenario is shown in **Figure 3** right.

Once we have managed to divide our polygon into smaller polygons via a diagonal, we can continue this process on each of the resulting polygons until only triangles are remaining. □

Now that we have successfully triangulated our polygon, we can observe our polygon as a graph, with all original vertices as nodes and all sides (as well as the newly drawn diagonals) as edges. We shall show that this graph is 3-colorable, or that each node can be assigned one of three colors such that no edge has two nodes of the same color.

We prove that our graph is 3-colorable by induction. First, the case where we have three nodes connected to each other (a triangle) is trivial. For any graph with more than 3 vertices, we can simply choose a diagonal edge and split the graph into two smaller graphs along it, while making sure both new graphs retain the diagonal edge. Now we color both smaller

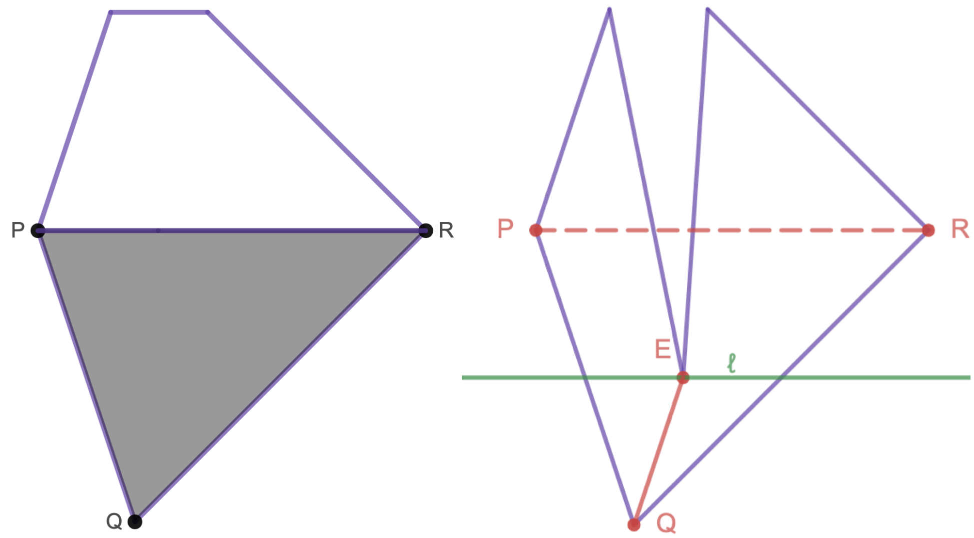

Figure 3. In the diagram at left (case 1), taking triangle $\triangle PRQ$ will suffice. However, in the diagram at right (case 2), the dashed red segment $\overline{PR}$ is no longer a valid diagonal. Instead, by dragging line ℓ downwards, we can construct a new red segment $\overline{QE}$ as a valid diagonal.

graphs using 3 colors only, which we can do by strong induction. We can then combine the graphs together if the shared edge has a similar coloring for both graphs. If not, we can simply switch around the node colors of the second graph using a permutation of the original configuration. Hence, we have a 3-coloring of our original graph. A coloring process of an example graph is shown in **Figure 4**.

Next, we note the following lemma which allows us to see why the $\lfloor \frac{n}{3} \rfloor$ was necessary in the original formula.

Lemma 2.2. *If a graph with n nodes is k-colorable, every coloring using k colors of the graph will contain at least one color that only appears at $\lfloor \frac{n}{k} \rfloor$ nodes or less.*

Proof. We simply apply the Pigeonhole Principle, setting the number of vertices equal to the number of pigeons and the number of colors equal to the number of holes. □

Using the lemma we see that when we color our polygon graph with 3 colors, at least one color will appear at at most $\lfloor \frac{n}{3} \rfloor$ nodes. Hence, placing a guard at every node of the least common color requires at most $\lfloor \frac{n}{3} \rfloor$ guards. Since every triangle in our triangulation has exactly one node of that specific color, the guard stationed at that colored node will be able to guard that specific triangle. Hence, we have found an arrangement of at most $\lfloor \frac{n}{3} \rfloor$ guards for a museum with n walls, proving **Theorem 1.3**.

3. Other Museums

Now that we have proved the main theorem regarding museum security, let us turn to some variants of this problem.

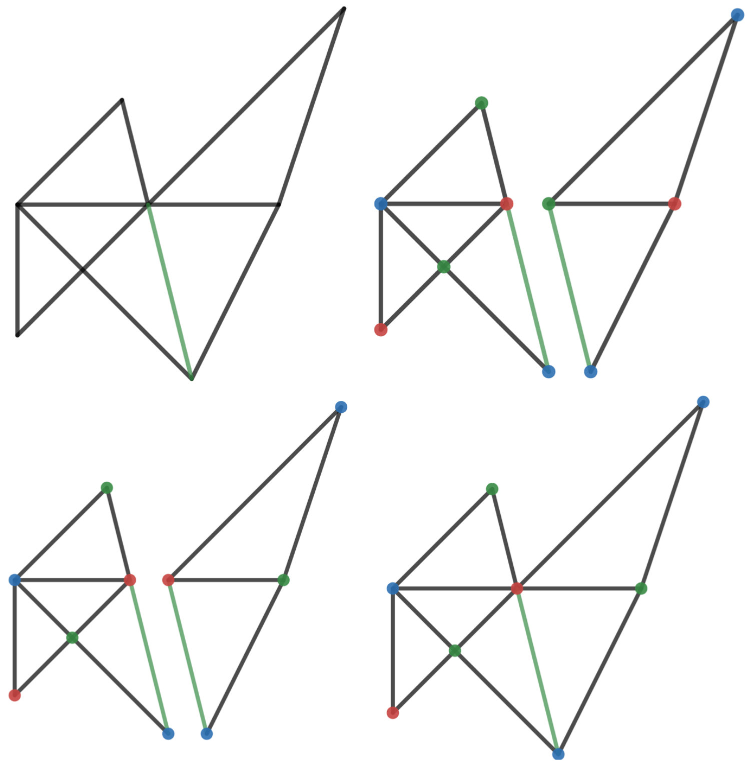

Figure 4. After choosing a diagonal to divide our graph by in the first figure, we have colored each of the two split graphs with 3 colors. Next we switch the smaller split graph's red nodes with green nodes in order to match the coloring of the green edge. Finally, we can combine the two split graphs into the fully 3-colored original graph.

Theorem 3.1. *The number of guards required to keep safe a museum with n orthogonal walls safe is at most $\left\lfloor \frac{n}{4} \right\rfloor$.*

There are three different proofs of this theorem: one from Kahn, Klawe, and Kleitman, another from Lubiw, and a third from Sack and Toussaint. While this also seems like a simple theorem to prove, all three proofs are much more challenging and are hence not elegant enough to be found in the Book. All three distinct proofs can be found at [O'r87].

Another variant of the problem asks for the minimum number of guards required to guard the museum if each guard can only be placed on a wall. The guard can move freely along eir wall and guard anything that can be observed when standing on any point on eir wall. It is known that at least $\left\lfloor \frac{n}{4} \right\rfloor$ are necessary for every possible museum (in fact, there is a generalized polygon that shows the necessity of m guards when there are $4m$ walls), but a proof for sufficiency is yet to be discovered.

While museums are fun to guard, we should also consider the consequences of an intruder invading the museum with a large amount of troops. This causes us to ponder the minimum number of guards stationed outside the museum to be able to see everything going on outside. In this case, many more guards are needed - for an n walled museum, up to $\left\lceil \frac{n}{2} \right\rceil$ guards

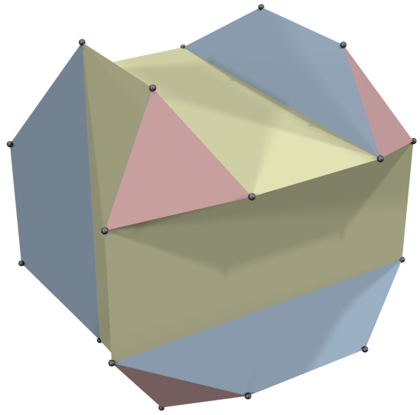

Figure 5. Placing a guard at each vertex is not enough!

may be required (note the ceiling, not floor function, that is used).

Notice that when we proved the formula for an arbitrary museum, we found a way to place every guard on a vertex and not in the interior of the museum. Doing the same for a 3-dimensional polyhedron, however, will not work. **Figure 5** shows an example of a polyhedron that cannot be fully guarded even if every vertex is occupied by a guard.

Hopefully during your museum job interview, the employers were not *too* unsatisfied with your 'nonsense' (it probably would have been enough if you simply cited the Book). If they were sufficiently impressed, you may even be able to land the job of gallery supervisor!

References

[AZHE10] Martin Aigner, Günter M Ziegler, Karl H Hofmann, and Paul Erdos. *Proofs from the Book*, volume 274. Springer, 2010.

[O'r87] Joseph O'rourke. *Art gallery theorems and algorithms*, volume 57. Oxford University Press Oxford, 1987.