FIVE-COLORING PLANE GRAPHS

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Abstract. In this expository paper, we will investigate plane graphs and their colorings. We will start off by proving that that any graph must be 6-colorable using Euler's formula. We will then improve to show that any graph must be 5-colorable. After this, we will talk about list chromatic numbers and prove that all planar graphs can be 5-list colored.

1. Introduction

Let's imagine the map of the United States with the state borders outlined. What is the minimum number of colors it would take to color each state such that no two neighboring states have the same color? In general, for any plane map, how many colors would it take color each region such that every bordering region is colored with a different color? We can investigate this problem further using a planar graph to represent the planar map. Before doing that we must first define what a graph is and what the coloring of a graph means.

Definition 1. A *graph* is an object that contains 2 sets: its vertex set and its edge set. The vertex set must be finite and nonempty, while the edge set can be empty. The elements of the edge set are two-element subsets of the vertex set.

Example 2. For example, we can construct a graph with the vertex set $V = \{A, B, C, D, E, F\}$ and the edge set $\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, E\}, \{C, F\}\}$. A visual representation can be seen in figure 1.

Definition 3. A graph is *colored* if each vertex is assigned a color such that no 2 adjacent vertices have the same color.

Each region on the plane map can be represented by a vertex and the vertices of the states that border each other can be connected with an edge. We can color the vertices to represent coloring a state. We will assume that the planar graph doesn't have any loops or multiple edges. A representation of this is shown in figure 2.

In 1852, it was conjectured that it would take up to only 4 colors to color any planar graph such that any two adjacent vertices cannot have the same color. This was eventually proven in 1976 by Kenneth Appel and Wolfgang Haken using the aid of a computer. We won't be discussing the 4-color theorem proof here because its proof is beyond the scope of this paper. Instead, we shall prove a weaker result: the 5-color theorem.

2. The 5-color Theorem

Definition 4. The *chromatic number* of a graph G $\chi(G)$ is the smallest number of colors with which a graph can be properly colored, meaning that no 2 adjacent vertices have the same color.

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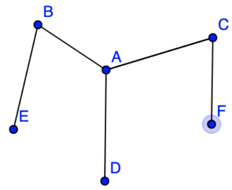

FIGURE 1. A basic example of a graph.

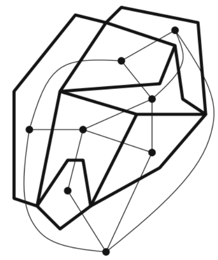

FIGURE 2. An example of a planar graph constructed from a planar map. (Proofs from The BOOK)

Unlike the 4-color theorem, we can prove that any planar graph can be colored with 5 or less colors without the assistance of a computer. But before we do that, we shall prove an even weaker result: the 6-color theorem. To do this, we'll utilize Euler's formula.

Theorem 5. *(Euler's Formula) If G is a connected plane graph with n vertices, e edges and f faces, then*

$$n - e + f = 2.$$

But before we prove Euler's formula, we must prove a lemma that will help us.

Lemma 6. *Euler's formula holds true for any tree T.*

Proof. We will use proof by induction. We know that a tree only contains 1 face, so Euler's formula for a tree reduces to $e = v - 1$. For the base case, we know that if the graph contains no edges and only 1 vertex, then the formula holds true. For the induction step, we assume that the formula works up to n vertices. Let T be be a tree with $n + 1$ vertices. We have to show that T contains N edges. Since T is a tree, it must have a vertex with a degree of 1. Let that vertex be k. If we remove k and the edge connecting it, we make a new tree T' with n vertices. By the inductive hypothesis, T' has n edges, meaning that T has n edges. Thus, Euler's formula holds for T. ■

Now that we've proven this lemma, we can prove Euler's formula.

Proof. We'll use proof by induction on the number of edges in the graph. For the base case, we have $e = 0$, which means that the graph has a one vertex with one region surrounding it. This means that we get $1 - 0 + 2$. Now for the induction step, let's suppose that this formula works for all graphs with n or less edges. Let's declare G to be a graph with $n + 1$ edges. We have two possible cases: G doesn't contain a cycle or it contains at least 1 cycle. If G doesn't contain a cycle, the graph is a tree, and we know that the formula works for trees because of Lemma 4. If G contains at least 1 cycle, let's pick an edge c such that it's on a cycle. Let's then remove c, forming a new graph G'. By removing c, we know that G' contains 1 fewer regions than G. By the inductive hypothesis, the formula works for G', meaning that $v' - e' + f' = .2$. We also know that $v' = v$, $e' = e - 1$, and $f' = f - 1$. By substitution, we get $v - e + f = 2$, thus proving Euler's formula. ■

Now that we have proven Euler's formula, let's see how we can apply it to prove the 6-color theorem.

Theorem 7. *Every plane graph G is 6-colorable such that $\chi(G) \leq 5$.*

Proof. We will do a proof by induction on n vertices. For $n \leq 6$ it's obvious. We know that G has a vertex v of degree at most 5. If we delete v and the edges connected to it, the resulting graph G' is a plane graph with $n-1$ vertices. By induction, G' can be 6-colored. Since v has up to 5 neighbors, at most 5 colors can be used to color these neighbors in the coloring of G. Thus, we can extend any G' 6-coloring to a 6-coloring of G by coloring v a color which hasn't been used by any of the 5 neighboring vertices. Thus, this proves that any graph G is 6-colorable. ■

Now that we have proven the 6-color theorem, let's try to prove a stronger result: that it only takes up to 5 colors instead of 6.

Theorem 8. *Every plane graph G is 5-colorable such that $\chi(G) \leq 5$.*

Proof. We will do a prove by induction on v, the number of vertices of G. For the base case $v = 1$, it can obviously be colored with ≤ 5 colors. In fact, this is the case for $v = 2, 3, 4, 5$ as well. We assume that for $v = n$ vertices, it can be colored in 5 or less colors. Let G be a graph with $k+1$ vertices. We know that G has a vertex A such that it has a degree of at most 5. Let's remove vertex A and the edges connecting it. We will call this new graph G'. This graph contains k vertices, meaning that it can colored with 5 or less colors. Suppose that we color G' in some way. We then have three possible cases.

Case 1: G' can be colored with less than 5 colors. This means that it can be colored with at most 4 colors. So when we put vertex A back with its edges, we can simply color A a new color that hasn't been used for G'. Therefore, G is 5-colorable.

Case 2: G' can be colored with 5 colors and the degree of vertex A is less than 5. In this case, vertex A is adjacent to at most 4 other vertices. Thus, when we put back A and its edges, we can simply color it the a color that hasn't been used by its 4 neighbors but has been used elsewhere in the graph. This means that G is 5-colorable.

Case 3: G' can be colored with 5 colors and the degree of vertex A is 5. Let P, Q, R, S, and T be the vertices adjacent to A. If any of these vertices are colored with the same color, then there's an open color for A, thus making G 5-colorable. However, if that's not the case, then we must try to recolor one of the vertices adjacent to A without upsetting the coloring of G'. Let's say that the colors for the adjacent vertices are 1, 2, 3, 4, and 5. This leads us to two possible sub-cases.
Sub-case i: There isn't a walk joining P and R consisting only of vertices colored 1 or 3. Figure 3 gives a picture of what part of G might look like. Let's recolor P with color 3 and interchange all of the colors of the vertices of all color 1 and 3 walks touching P as shown in figure 4. R and its color 1 and 3 walks remain unaffected. This leaves color 1 freed for vertex A. Thus, G is 5-colorable.
Sub-case ii: There is a walk joining P and R consisting only of vertices colored 1 or 3. Figure 5 shows an image of what a part of G could look like. In the figure, we assume that all walks that touch either Q or S and consisting entirely of vertices colored 2 or 4. Also, there's no color 2 and 4 walk joining Q and S. Since G is planar, there aren't any edge crossings. Vertex Q is surrounded by a cyclic sub-graph of P, U, W, X, R, and P. None of those vertices is colored either 2 or 4. The other vertices are colored 1 or 3. This means that any

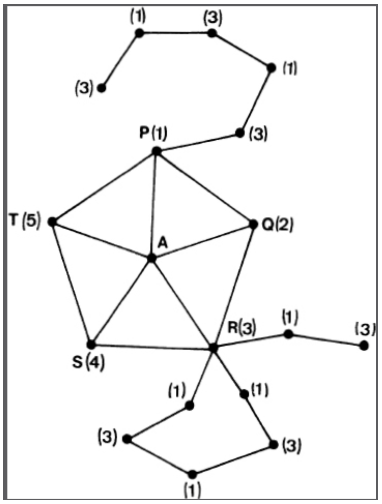

FIGURE 3. (Introduction to Graph Theory)

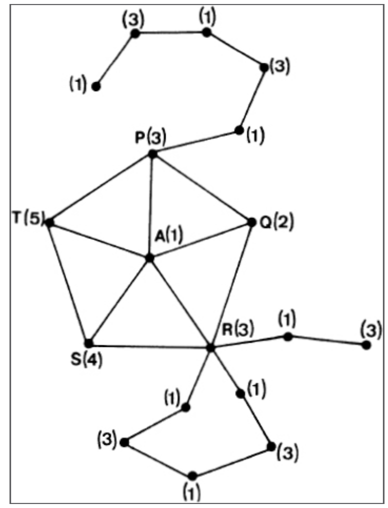

FIGURE 4. (Introduction to Graph Theory)

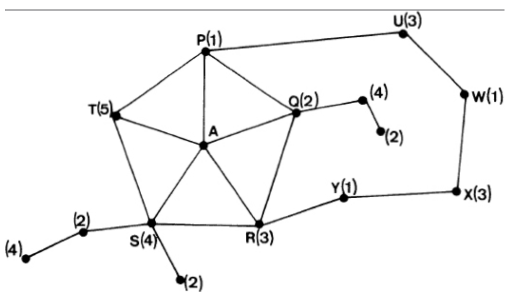

FIGURE 5. (Introduction to Graph Theory)

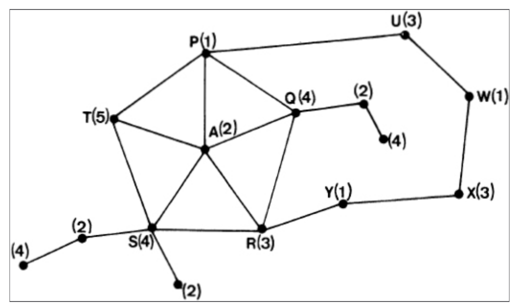

FIGURE 6. (Introduction to Graph Theory)

color 2 and 4 walk from Q to S has to pass through one of the vertices of the cyclic graph. But since none of the vertices in the cyclic graph are colored 2 or 4, there must be no walk in the first place. We can now interchange the colors of the color 2 and 4 walks touching Q. Vertex S is unaffected, so A can be colored with color 2. This means that G can be colored with 5 colors. ■

3. THE LIST CHROMATIC NUMBER OF PLANE GRAPHS

Let's now talk about the list chromatic number of a plane graph. But before we do that, let us define what a list coloring is and what a list chromatic number is

Definition 9. A *list coloring* is a coloring $c : V \longrightarrow \bigcup_{v \in V} C(v)$ where $c(v) \in C(v)$ for each $v \in V$.

Definition 10. A *list chromatic number* $\chi_l(G)$ is the smallest number k such for any list of color sets $C(v)$ with $|C(v)| = k$ for all $v \in V$ there always exists a list coloring.

In 1979, Erdös, Rubin, and Taylor conjectured that for every graph G, $\chi_l(G) \leq 5$. This was later proven by Carsten Thomassen.

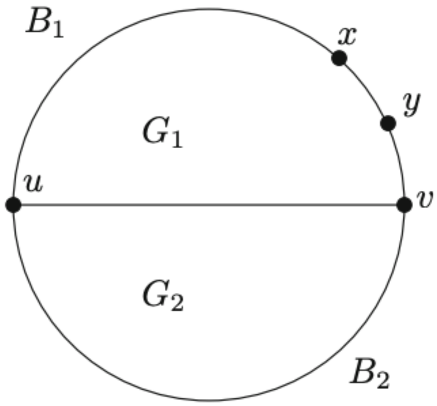

FIGURE 7. (Proofs from The BOOK)

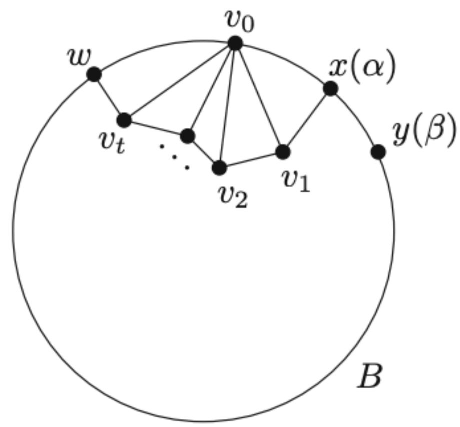

FIGURE 8. (Proofs from The BOOK)

Theorem 11. *All planar graphs G can be 5-list colored: $\chi_l(G) \leq 5$.*

Proof. Let $G = (V, E)$ be a near-triangulated graph, and let B be the cycle that bounds outer region. For $C(v), v \in V$:, we assume that 2 adjacent vertices x, y of B are already colored with different colors α and β. We also assume that $|C(v)| \geq 3$ for all other vertices v of B and that $|C(v)| \geq 5$ for all vertices v in the interior.

We can then extend the coloring of x, y to a proper coloring of G by choosing colors that are from the lists. In particular, $\chi_\ell(G) \leq 5$

This is obvious for $|V| = 3$ because for the only uncolored vertex v we have $|C(v)| \geq 3$, meaning that there is a color available. We can now use induction.
Case 1: Suppose B has a chord, an edge that isn't in B that joins two vertices $u, v \in B$. The sub-graph G_1 is bounded by $B_1 \cup \{uv\}$ and contains x, y, u and v is near-triangulated and therefore has a 5-list coloring by induction. Suppose that in this coloring the vertices u and v are colored γ and δ. We then look at the bottom part G_2 bounded by B_2 and uv. Regarding u, v as pre-colored, we see that the induction hypotheses are also satisfied for G_2. Hence G_2 can be 5 -list colored with the available colors, and thus the same is true for G.
Case 2: Suppose B doesn't have a chord. Let v_0 be the vertex on the other side of the α -colored vertex x on B, and let $x, v_1, \ldots, v_t, w$ be the neighbors of v_0 since G is near-triangulated we have the situation shown in the figure. Construct the near-triangulated graph $G' = G \backslash v_0$ by deleting the vertex v_0 from G and all edges connected to v_0. This G' has an outer boundary $B' = (B \backslash v_0) \cup \{v_1, \ldots, v_t\}$. since $|C(v_0)| \geq 3$ by our second assumption, that $|C(v)| \geq 3$ for all other vertices v of B, there exist two colors γ, δ in $C(v_0)$ different from α. We then replace every color set $C(v_i)$ by $C(v_i) \backslash \{\gamma, \delta\}$, keeping the original color sets for all of the other vertices in G'. Then G' satisfies all assumptions and is thus 5-list colorable by induction. Choosing γ or δ for v_0, different from the color of w, we can extend the list coloring of G' to all of G. ■

In addition to the 5-list color theorem, there was a conjecture that claimed something even stronger.

Conjecture 12. $\chi_l(G) \leq \chi(G) + 1$, *where $\chi(G)$ is the ordinary chromatic number.*

This conjecture actually ended up being false. We know that the ordinary chromatic number is less than or equal to 4 for all planar graphs G. This leads to three cases. Either we have $\chi(G) = 2$ and $\chi_l(G) \leq 3$; $\chi(G) = 3$ and $\chi_l(G) \leq 4$; or $\chi(G) = 4$ and $\chi_l(G) \leq 5$. The conjecture ends up failing at the second case. This was initially shown by Margit Voigt using a graph constructed by Shai Gutner. The graph contains 130 vertices.

References

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