The Rectangular Peg Problem

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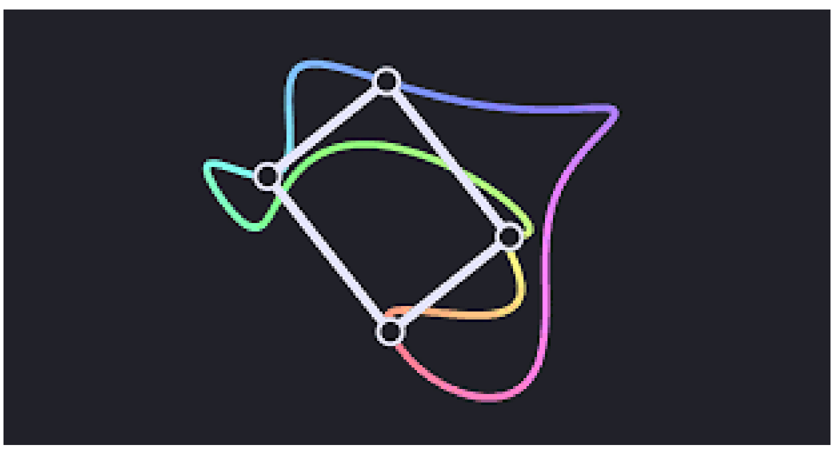
1 Introduction

1.1 The Problem

Theorem
1.1. *Every simple(i.e.the loop cannot intersect itself) continuous closed loop always has four points such that if connected they form a rectangle*

For example, look at the picture given below



1.2 An Informal Intro To Topology

Note by no means is this a proper holistic definition of the subject. I will informally explain only the relevant parts.

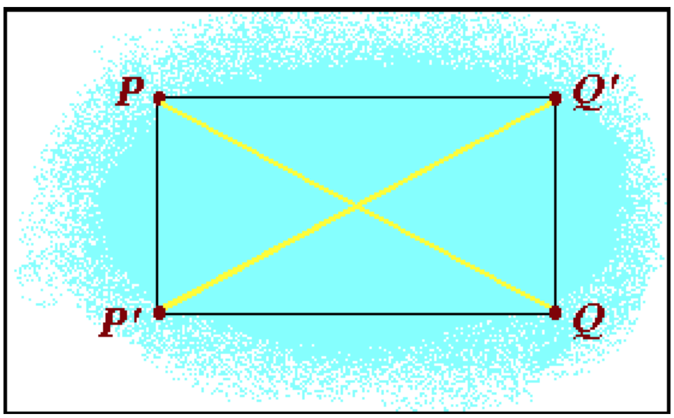
Topology is a study of curves and spaces. It focuses on the properties of shapes which remain constant under deformations twists stretches, however, cutting and gluing are not allowed.
Imagine a rubber band, but you can stretch it as much as you want. It does not tear.

One important consequence is that length plays no role here as shapes can be squished or expanded.
A good example of this would be say taking a circle and stretching it into an ellipse or vice versa.
Using some geometry might help for gaining an intuition in a few special cases but we need a more general approach.

2 The Solution

2.1 Rectangle Properties

The diagonals of any rectangle bisect each other and are equal.

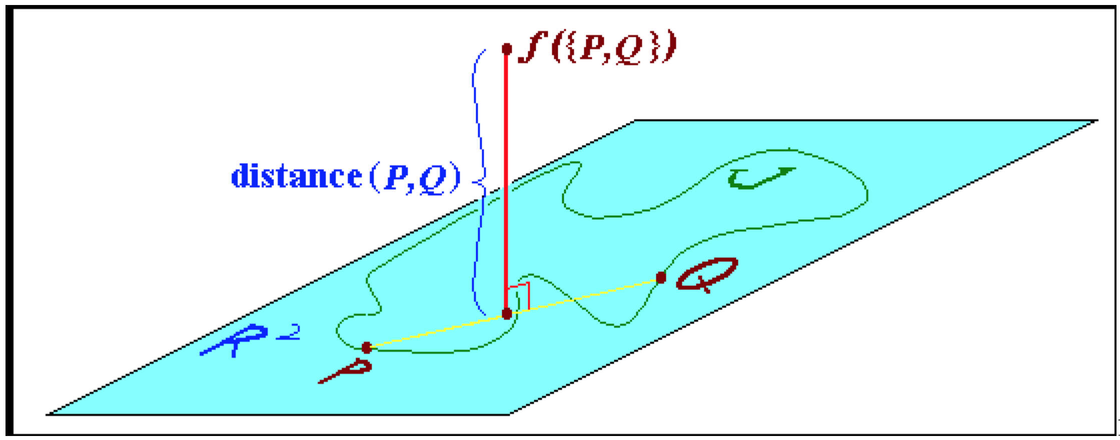


Lastly,as per our previous definition of topology, a rectangle can be squished/expanded into a circle.

2.2 Mapping The Curve Into 3d Space

Definition
2.1. *We define a function $f : x- > \mathbb{R}$ which maps our $2d$ curve (say j) into $3d$ space.$f(P, Q)$ is the point in $\mathbb{R}^3$ lying directly over the midpoint of the segment PQ but with z-coordinate equal to the distance from P to Q.*

In essence, we place our curve on the xy plane then fix 1 point on the curve and keep on slowly moving the 2nd one starting on the first to away from it and gradually back to it. Each time, we plot a line between these two points and plot a point directly above the midpoint of the line such that the height is equal to the distance between the points.



[1]

Remark
1. *As the point Q approaches point P, the plot on the z-axis gets less and less and when they are very close the value is almost 0.This means that the base of the mapping is the curve itself. Also,the function is continuous.*

Corollary
2.0.1. *If four points exist on the curve such that if we connect them in a specific way, they form equally long lines which have the same midpoint each other then they form a rectangle*

According to our function that would mean that there are two pairs of points which map onto the same point in 3d space (since they have the same length and midpoints which bisect the lines). The above statement is as good as saying that the function intersects itself (because it is continuous).

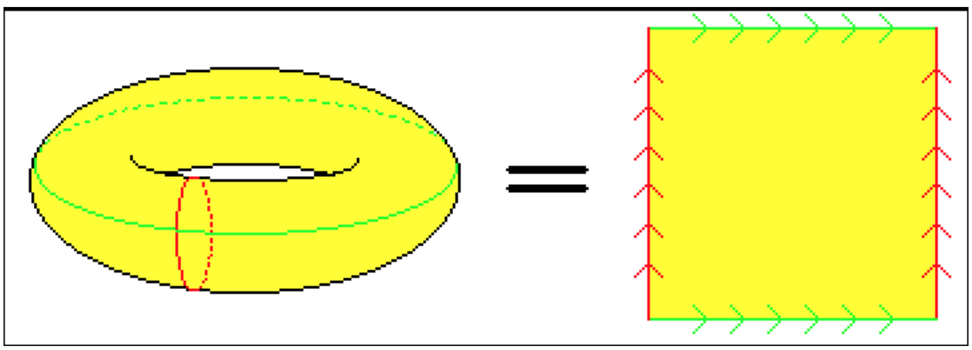
2.3 Simplifying Pairs Of Points

Lemma
2.1. *A pair of points on our curve corresponds uniquely to a point on a torus*

Proof. The curve is topologically equivalent to a circle, so topologically the cartesian product $\mathbf{j} \times \mathbf{j} = \mathbf{c} \times \mathbf{c}$... A torus! A pair of points on our loop would thus correspond to a unique point on the torus.
Another way to understand the same would be to snip the curve at any point.Flatten it out into a line(remember the ends of the line correspond to the same point where the cut was made otherwise every point corresponds uniquely to another)

of size 1 so finding a pair of points on the curve is now as good as a pair of points on the number line - the coordinate plane.
The two points are now two lines: the x-axis and the line $y = 1$ on the top and the y-axis and the line $x = 1$ on the right. These pairs of lines are the same and in order to be consistent with our definition of each point corresponding to uniquely a pair of points, we need to glue these. We first glue either the lines parallel to the y or x-axis. This gives us a pipe. Now we must glue opposite ends of the pipe (the other line and the axis). This forms the torus we were looking for. Each point on this torus corresponds uniquely to a pair of points on the curve. Note that a torus is continuous.

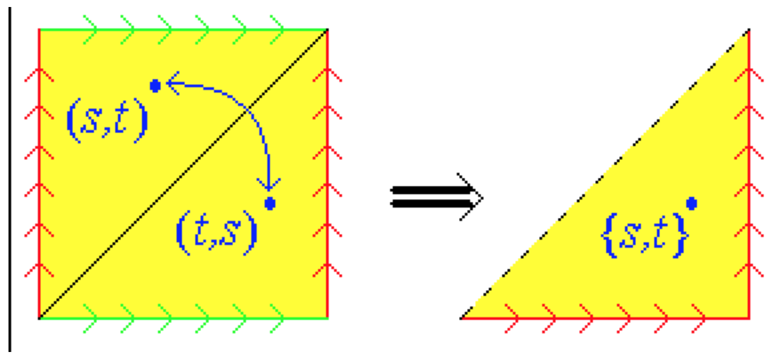
□



2.4 A Small Mistake

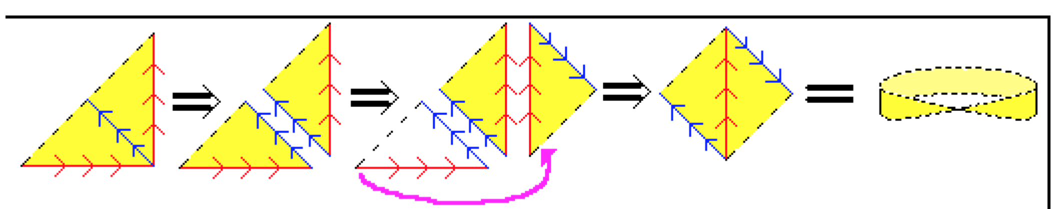
Remark
2. *There is a small problem with our approach.*
The points P and Q are different than Q and P (they form ordered pairs). This brings the notion of a first point even though there is none, and a trivial solution which is wrong: the pair of points have the same length, midpoint, and distance as the same pair so the line between the points is a rectangle. To undo this error we must consider unordered pairs of points.

In order to facilitate this change, we need to make the point (x, y) the same as the point (y, x). Our first thoughts jump to symmetry about the line $y = x$ and that is what we need to do. We must fold our square over the line $y =$x and that problem is solved.



Lemma
2.2. *A pair of unordered points on our curve corresponds uniquely to a point on a Möbius strip.*

Proof. We did solve a problem, but in doing so created a new one. Our old trick of gluing the sides to form the torus will not work as we now deal with an isosceles right angled triangle.
In the triangle, we need to glue the base and the height in such a way that the point touching the hypotenuse on the base touches the opposite point on the height and vice versa. This one is not so obvious but we are saved by a stroke of genius. The trick is to cut the triangle through the middle splitting the right angle into two $\frac{\pi}{4}$ angles forming two triangles.
Rotate the bottom triangle by $3\frac{\pi}{2}$. The two sides are now fairly straight forward to glue leaving us with a square with two sides to glue. Unlike our approach in the torus where $(0, y)$ would correspond to $(1, y)$, here $(0, y)$ would correspond to $(1, 1 - y)$.
Twisting that square in 3d space yields a Möbius strip! Therefore an unordered pair of points on any loop uniquely corresponds to a point on the Möbius strip (an open Möbius strip to be precise).



□

2.5 Completing The Proof

Lemma
2.3. *All we need to do is show that is impossible to place that Möbius strip on the 3d mapping defined earlier in a one-one fashion.*

Proof. The image set $f(X)$ must have some collisions where more than one point of X is carried to the same point of $\mathbb{R}^3$. a topologist might require more convincing, but would believe it as soon as you remind them that you can't embed the space you get by sewing a disk to a Möbius strip (the projective plane) into three dimensions.
This completes our proof. □

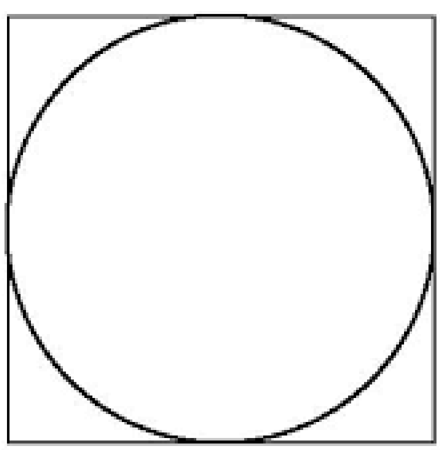
3 Extensions Of The Problem

3.1 Recent Work On It

A recent paper of C. Hugelmeyer combines Vaughan's theory with modern knot theory results to show that the curve always has an inscribed rectangle of aspect ratio $\sqrt{3}$.// During the pandemic, two mathematicians have worked to prove the same for smooth Jordan curves.www.paperUDur.com

3.2 An Unsolved Extension

The problem we just completed was a loose and simpler version of the Toeplitz Conjecture. The question remains the same except this time instead of a rectangle, the shape is a square.



That makes it considerably tougher (or well hidden) as a solution has not yet been found!

4 After-thoughts

Please note that this is not definitive or necessarily right. It's my take on the intuition behind the proof.

The question bears a striking resemblance to a crossover between topology and geometry and so I feel that topology was its natural way of course.
As for the proof, I personally feel the proof was thought of in a way slightly different than what I presented. I will be listing my intuition behind it in the order I feel it was thought of.
Thinking of a rectangles as a pair of points is not something out of the ordinary and I feel this would have been the obvious choice since taking side lengths would be too tough. This is for general curves and that approach would not get very far.
Next, having pairs of points simplified it into a torus was a clever move and expressing pairs of points is not something new to us. Take for example the xy plane which is nothing but a representation of points on the number line. When doing so, we had points on a 1d line mapped on to a 2d plane. Analogically, points on the 2d curve were mapped in a 3d figure. Torus is a very friendly and clean choice for it as circles are good to work with and it has many other properties like being continuous (this is a key factor as if we chose something discrete it would mean a small nudge there may not have been defined on the actual curve which would not make sense), being friendly to direction (unlike say a Möbius strip we can rigorously define vectors here, however, this property does not help us here).

It is fairly common for mathematicians to think of a torus as a folded square. So the idea of folding the square along the line $y = x$ to find a mapping for unordered pairs was somewhat natural. For the next step, I would highly recommend cutting an isosceles right triangle drawing a few arrows on it and trying it yourself. I played around with the cut out for some time. Unlike what is presented in the proof, I simply imagined stretching the base of the triangle into forming a trapezoid. Then if you try gluing the sides in 3d space just the way you would form a torus, you would notice the difference in orientation of the arrows. To undo that we give it a twist and get a Möbius strip. When doing so I felt a bit awkward considering a Möbius strip is non-orientable (we cannot rigorously define directions here. For an intuitive visual proof check www.Möbiusstrip.com)
If you have the same thought think about the step just before and realise it is redundant.
Since we already know that pairs of points are depicted by the Möbius strip all that remains to do is show that there are two distinct points on the Möbius

strip which correspond to points which satisfy our criteria for a rectangle i.e. they form a pair of lines of equal length which bisect each other.
If we could somehow make a function that would take points on the strip and return the distance between those points and the midpoint then we would just need to show that the function is not injective(one-one).
I feel that this was a good initial intuition and the next step was just improvising on this thought. The mapping of the function into 3d space was the improvisation required. From our initial thought, we needed one thing which could represent the midpoint of a line and its distance. The point that distance above the midpoint does just that trick(this time we needed the midpoint as an ordered pair obviously so taking something like the $abscissa \times ordinate$ would simply not have worked as say 3×8would be the same as 8×3 but $(3, 8)$ and $(8, 3)$ are obviously different). So now we can imagine a function that takes a point on the Möbius strip and returns a point in 3d space that is above the midpoint of the line corresponding to that point on the strip at a height equal to the distance of the Möbius strip. We need to show that the functions is not injective. The Möbius strip here is our domain and the points in 3d space our range. If we simply put the Möbius strip on the output and find two points on the Möbius strip landing on the same point we can conclude that the function is many-one. In other words, the strip must intersect itself to form that output.
The most difficult and creative parts of our proof are already over. The next part is fairly straightforward and that completes our proof.

5 Acknowledgements

This was based on a rather dense paragraph (towards the very end) in a paper by Mark Meyers www.thepaper.com) which is inspired by a lecture by H Vaughan at Univ. of Ill. at Urbana, 1977.
another inspiration was 3blue1brown (a famous Youtuber who takes an intuitive and visual approach to lots of basic concepts of math and cs) www.3b1bvid.com
Another source of the material and most of the pictures was a paper from the University of Idaho www.paperUIdaho.com
A paper by Josh Greene at Boston University gave a lot of useful background www.paperBU.com
I got to know of topology through a course offered by a local university on Youtube www.topologyvid.com albeit the English the professor speaks is hard to understand
The extensions to this were mainly taken from a paper by R Schwartz www.BrownUpaper.com
The informal visual proof for non-orientability of the Möbius strip was taken from the site, mathinsight.
The result of the recent work was taken from a paper by Joshua Evan Greene and Andrew Lobb www.paperUDur.com
The very first picture was taken from an article by quantamagazine www.qmag.com