On the Mathematics of Bracelets

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Abstract. We discuss knot theory from the perspective of bracelets, starting with Brunnian "Rubberband" links and similar Brunnian links inspired from bracelets. We then analyze braids and their relationship with knots, ending with a knot invariant. Finally, we reference two open problems related to the proof that Brunnian links cannot be made from circles.

1 Introduction

We provide some preliminaries:

Definition 1.1. A *knot* is an injection from the unit circle $\mathbb{S}^1$ to $\mathbb{R}^3$.

Definition 1.2. A *link* is a finite collecion of nonintersecting knots. Each knot in a link is called a *component*.

It is difficult to analyze topological objects in $\mathbb{R}^3$, so we make diagrams of them in the plane. To define a notion of equivalence between diagrams, we need the following theorem due to Reidemeister (see [vR]):

Theorem 1.3. *Two knot or link diagrams are equivalent if one can be transformed into the other through a series of deformations and Reidemeister moves.*

The three Reidemeister moves are displayed visually (Figures 1.1, 1.2, and 1.3). Using this definition of equality, we find some trivial cases of knots and links.

Definition 1.4. An *unknot* is any knot whose diagram is equivalent to a circle in $\mathbb{R}^3$.

Figure 1.1: Reidemeister 1.

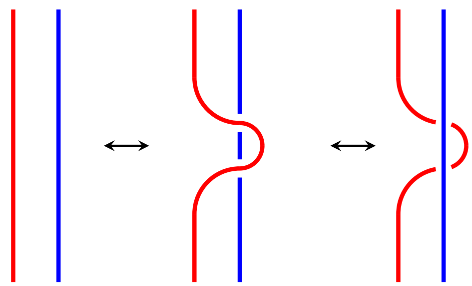

Figure 1.2: Reidemeister 2.

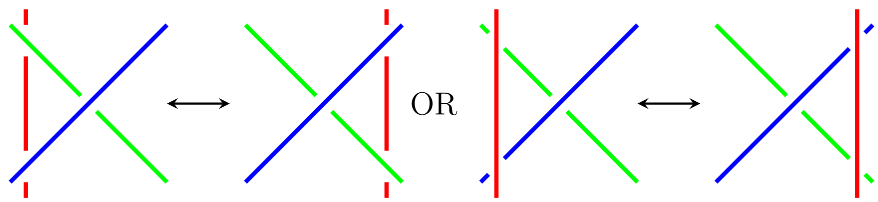

Figure 1.3: Reidemeister 3.

Definition 1.5. A link is *unlinked* if its components can be separated through continuous deformations without any intersection between components.

Invariants over equivalent 2-dimensional diagrams are also a useful tool to identify knots or links, such as the Jones polynomial.?????

Definition 1.6. A *Brunnian link* is a link such that removing any of its components results in an unlinked set of unknots.

One noteworthy example of a Brunnian link are the Borromean Rings (Figure 1.4).

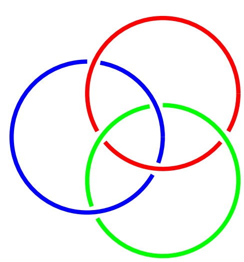

Figure 1.4: The Borromean Rings.

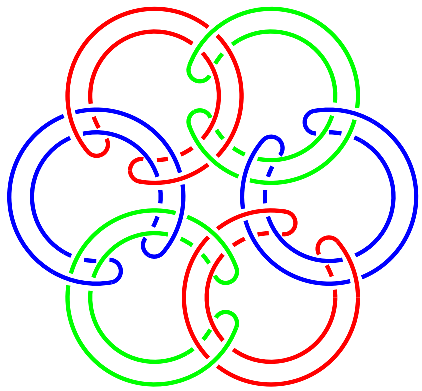

Figure 2.1: Rubberband links with 6 components.

2 Rubberband Links

Theorem 2.1. *There are Brunnian links with n components for all $n > 2$.*

This can be shown quite effortlessly with rubberband links (see Figure 2.1). Each component is a deformed unknot, and it can be verified that removing any component results in unlinked unknots. The nature of rubberband links allows us to add as many components as we like while keeping them a Brunnian link (see Figure 2.2).

The primary inspiration for these links comes from Rainbown Loom, a rubber band bracelet toy, hence the name "rubberband". The simplest bracelet, known as single, is a sequence of rubberband links what can be made as long as desired. The algorithm for building the bracelet shows removing any component will undo it. (See `https://www.rainbowloompatterns.com/images/single.jpg` for a picture of single.)

The Jones polynomials of rubberband links are known, but they do not form any pattern as the number of components increases. Calculating them can be done with an algorithm devised by Kauffman which calculates the Jones polynomial of any knot.

Rainbow Loom also has many other bracelets, forming more sophisticated patterns of linked unknots that can repeat indefinitely. Fishtail, for example, is similar to the rubberband links, but with more tightly woven components. (See `https://culturela.org/wp-content/uploads/2017/05/RAINBOW-LOOM.jpg` for a picture of fishtail.)

3 Braids and Their Closures

Braids are also a common topological object in friendship bracelets. A *braid* is the tangling of a number of strands. The strands start out laid in a parallel

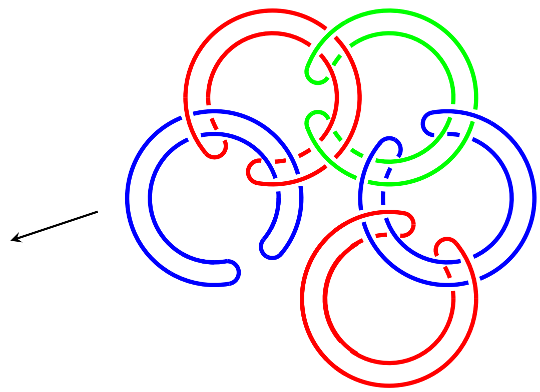

Figure 2.2: Removing a component allows us to extract unknots which are no longer connected to the link one at a time.

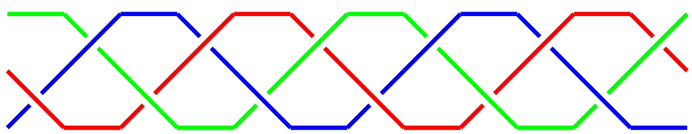

Figure 3.1: A diagram of the traditional braid.

fashion, and consecutive strands can then be crossed over or under each other in some order. No strand can be crossed with itself. To illustrate, the traditional braid, known best as a hairstyle, is displayed in Figure 3.1.

Braids can be closed by connecting their corresponding endpoints, forming knots or links (see Figure 3.2). This is called the *closure* of a braid. For example, the closure of shorter or longer segments of the traditional braid can form an unknot, the Hopf link, and the trefoil knot, whose braid is displayed in Figure 3.3.

Alexander ([Ale23]) was able to articulate an initial result on the relation between links and braids:

Theorem 3.1. *Every link can be formed by a braid closure.*

After this result, one may naturally wonder about the conversion from braids to knots or links and vice versa. Unsurprisingly, there are many, in fact infinitely many, braids that can form the same knot or link. However, this can be overcome without too much difficulty: we can select a particular braid to form a knot invariant.

Definition 3.2. Consider the set of braids whose closures form a specific knot or link and have the minimal number of strands. Then, the *minimum braid* of

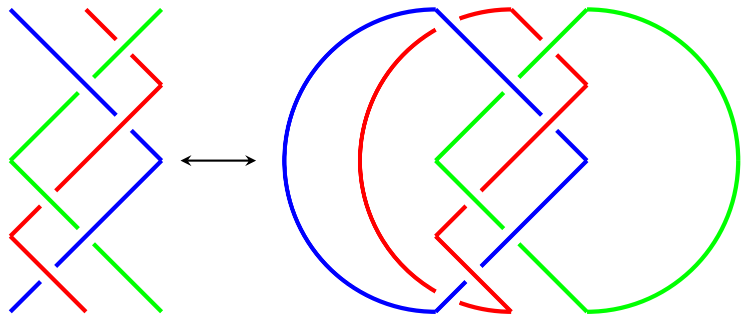

Figure 3.2: Closing this braid (by connecting corresponding endpoints) constructs the Borromean Rings.

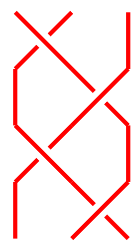

Figure 3.3: Closing this small traditional braid with four crossings constructs a trefoil knot. Colors are removed to show that only one component is formed.

the knot or link is the braid in that set with the minimum number of crossings between strands.

Gittings showed that the minimum braid is a complete invariant over all possible diagrams of a knot or link and provided an algorithm to generate the minimum braid of any diagram. [Git04] The general idea is to cut each of the knot components a number of times into strands and straighten out all the strands in the same direction so that no strand crosses itself. This creates a braid whose closure forms the knot or link in question, and we can use a series of equivalence rules (including braid equivalents of the Reidemeister moves) to transform this braid into the minimum braid. Since the minimum braid also provides an alternate way to visualize knots and links through their crossings, it helps in studying their properties.

Definition 3.3. The *braid index* of a knot or link is the minimum number of strands needed to form a braid closure of it.

The braid index, for instance, can be bounded for various classes of knots and gives one interpretation of how difficult a knot or link is to untie. Another more typical interpretation is the following:

Definition 3.4. The *unknotting number* of a knot is the minimum number of crossings that need to be reversed to make it an unknot.

The minimum braid is very helpful in determining the unknotting number because it provides a simple visualization of all the crossings and makes it easy to flip them.

4 Convex Curves

A well known fact about Brunnian links is that they cannot be constructed entirely out of circular components. [FS87] One may ask whether there are stronger characterizations of Brunnian links in $\mathbb{R}^3$.

Definition 4.1. A *convex* curve is a curve which bounds any convex region in $\mathbb{R}^2$. A *strictly convex* curve is a convex curve which does not contain any line segments.

All the Brunnian links we have discussed so far, except for the Borromean Rings, cannot be formed in $\mathbb{R}^3$ out of strictly convex curves. Other examples have led to speculation on whether the Borromean Rings are the only Brunnian link made out of strictly convex curves. Howards (see [How06]) was able to show that

Theorem 4.2. *The Borromean Rings are the only Brunnian links with 3 or 4 components made of strictly convex curves.*

The proof analyzes the cases of intersections in $\mathbb{R}^3$ between three or four planar discs, each formed by the curve of a component. In his master's thesis, Davis extended the proof to 5 components. [Dav05] However, the combinatorial nature of Howards' argument makes it difficult to generalize further. It remains an open problem whether there are any Brunnian links made of strictly convex curves with more than 5 components. An even stronger conjecture, also provided by Howards, is the following:

Conjecture 4.3. *The Borromean Rings are the only Brunnian links that have projections onto* $\mathbb{R}^2$ *made entirely out of strictly convex curves.*

This asks whether any other Brunnian links can be diagrammed in the plane using only strictly convex curves.

References

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