BUFFON'S NEEDLE PROBLEM

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1. Introduction

Buffon's needle problem was a question by mathematician Georges-Louis Leclerc, Comte de Buffon in the late 1700s, asking the following:

Question 1.1. *Suppose that you drop a short needle on ruled paper – what is then the probability that the needle comes to lie in a position where it crosses one of the lines?"*

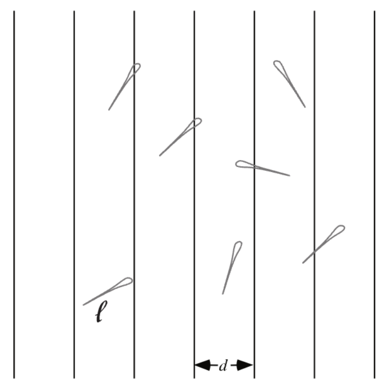

Figure 1. Image from Wolfram Math World

For this problem, we will assume that the needle is dropped randomly onto the paper. That is, the needle's center must have equal probability of being at any point on the paper and all possible angles that the needle can make with the horizontal has equal probability.

Theorem 1.2. *Consider a ruled paper with equally spaced lines at distance d from each other. For a short needle of length $l \leq d$ that is dropped on the ruled paper (as shown in Figure 1), the probability that it crosses one of the lines is*

$$p = \frac{2}{\pi}\frac{l}{d} \tag{1.1}$$

Proof. For a needle of any length, Let p_i be the probability that it hits exactly i different lines on the paper. Then, the probability that the needle will cross at least one of the lines is

$$p = p_1 + p_2 + p_3 + p_4 + ...$$

The expected value of the number of crossings is

$$E = p_1 + 2p_2 + 3p_3 + 4p_4 + ...$$

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For a short needle, it is impossible to get more than one crossing so $p_i = 0$ for all $i \geq 2$. Therefore, we get the expected value and the probability are the same or

$$E = p$$

It follows that to find the probability of a short needle hitting one of the lines, we need to find the expected number of crossings of the needle. Let $E(x)$ be the expected value of the number of crossings of a needle of length x. For a needle of length $x + y$, its expected value can be calculated by separating the needle into a front end of length x and back end of length y. Using linearity of expectation, we get

$$E(x + y) = E(x) + E(y)$$

Therefore, for integer x, $E(x) = cx$ where c is a constant equal to $E(1)$. We now want to compute $E(\frac{m}{n})$. Using our previous equation, we get

$$nE(\frac{m}{n}) = E(m)$$

$$E(m) = cm$$

Therefore,

$$E(\frac{m}{n}) = c\frac{m}{n}$$

This allows us find the expected value of the number of crossings for a needle of rational length.

We now want to find the value of c. Consider a needle in the shape of a perfect circle C of diameter d. This needle will always hit the lines on the paper at exactly two points. This circle can be approximated by an inscribed polygon P_n and a circumscribed polygon P^n such that

$$E(P_n) \leq E(C) \leq E(P^n)$$

Using our formula for expected value, we get

$$cl(P_n) \leq 2 \leq cl(P^n)$$

As $n \to \infty$, we get

$$\lim_{n\to\infty} l(P_n) = d\pi = \lim_{n\to\infty} l(P^n)$$

Therefore, the two polygons are good approximations for C. Additionally, for $n \to \infty$, we get

$$cd\pi \leq 2 \leq cd\pi$$

Therefore,

$$c = \frac{2}{\pi d}$$

$$p = \frac{2}{\pi}\frac{l}{d}$$

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Another way to obtain this result is to integrate over all possible angles that the needle can take on.

Proof. Let α, such that $0 \leq \alpha \leq \frac{\pi}{2}$, be the angle from the horizontal of the dropped needle. Then, the needle has height $l \sin \alpha$. For a given line, there is a distance of $l \sin \alpha$ where the needle will intersect the line out of a total distance of d, giving the probability $\frac{l \sin \alpha}{d}$ of hitting one of the lines. We can take the average over all α to get

$$p = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{l \sin \alpha}{d} d\alpha = \frac{2}{\pi} \frac{l}{d} \Big|_0^{\frac{\pi}{2}} = \frac{2}{\pi} \frac{l}{d}$$

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This method of integration using the slope of the needle allows us to generalize this formula to longer needles.

Theorem 1.3. *For a long needle $l \geq d$ that is dropped on a ruled paper with equally spaced lines at distance d from each other, the probability that it crosses one of the lines is*

$$p = 1 + \frac{2}{\pi} \left(\frac{l}{d} \left(1 - \sqrt{1 - \frac{d^2}{l^2}} \right) - \arcsin \frac{d}{l} \right) \tag{1.2}$$

Proof. For a longer needle, when $l \sin \alpha \leq d$, there is a $\frac{l \sin \alpha}{d}$ probability of crossing a line. For larger angles, the height of the needle will be greater than d so it must hit a line. Therefore,

$$\begin{aligned} p &= \frac{2}{\pi} \left(\int_0^{\arcsin \frac{d}{l}} \frac{l \sin \alpha}{d} d\alpha + \int_{\arcsin \frac{d}{l}}^{\frac{\pi}{2}} 1 d\alpha \right) \\ &= 1 + \frac{2}{\pi} \left(\frac{l}{d} \left(1 - \sqrt{1 - \frac{d^2}{l^2}} \right) - \arcsin \frac{d}{l} \right) \end{aligned}$$

Using Mathematica, we get the following probability graph where the x-axis represents the length of the needle, the y-axis represents the probability that the needle crosses a line, and $d = 1$.

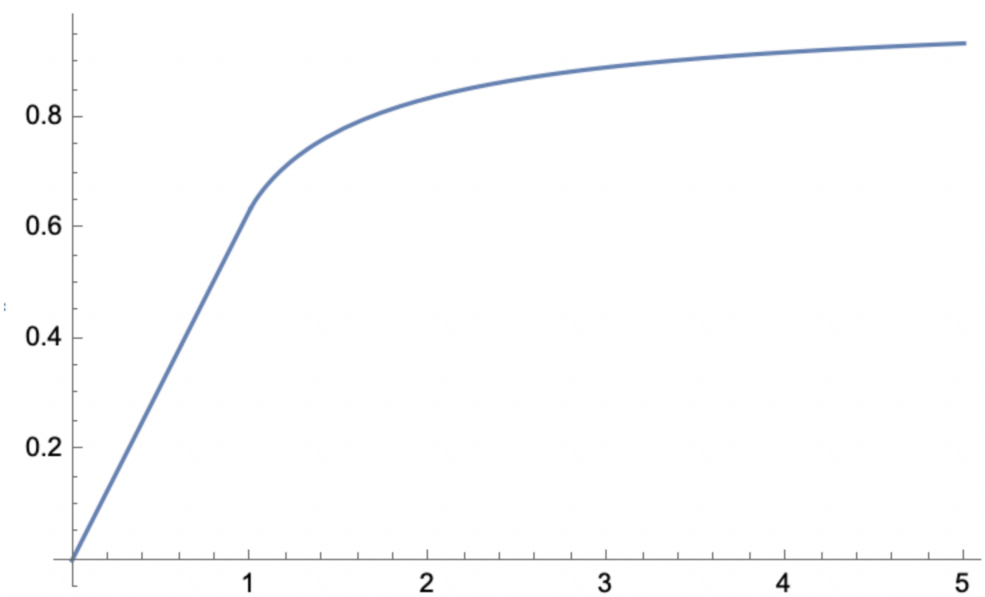

Figure 2. Length of Needle vs Probability of Crossing

We notice that when $l = d = 1$, $p = \frac{2}{\pi}$ and p approaches 1 as x approaches infinity. ∎

2. Buffon's Needle Problem for Convex Polygons

Buffon's needle problem can be generalized to other needles in the shape of a convex polygon.

Theorem 2.1. *For a thin plate in the shape of a polygon of perimeter P with diameter less than d, which is the distance between two adjacent lines on a ruled paper, the probability of crossing a line is*

$$p = \frac{P}{\pi d} \tag{2.1}$$

Proof. Consider a plate in the shape of a quadrilateral. Let A, B, C, and D be the sides of this quadrilateral. The plate can cross a line at either 0 or 2 points. Therefore, we get

$$p = P(AB) + P(AC) + P(AD) + P(BC) + P(BD) + P(CD)$$

where $P(XY)$ is the probability that the segments X and Y cross a line. We now look at the probability that each side hits the line based on the other sides that also hit the line. This gives us

$$P(A) = P(AB) + P(AC) + P(AD)$$
$$P(B) = P(BC) + P(BD) + P(BA)$$
$$P(C) = P(CD) + P(CA) + P(CB)$$
$$P(D) = P(DA) + P(DB) + P(DC)$$

Therefore,

$$P(A) + P(B) + P(C) + P(D) = 2p$$

Let $\alpha, \beta, \gamma, \delta$ be the lengths of A, B, C, D, E respectively. From Theorem 1.2, we get

$$P(A) = \frac{2}{\pi}\frac{\alpha}{d}, P(B) = \frac{2}{\pi}\frac{\beta}{d}, P(C) = \frac{2}{\pi}\frac{\gamma}{d}, P(D) = \frac{2}{\pi}\frac{\delta}{d}$$

Therefore,

$$p = \frac{\alpha + \beta + \gamma + \delta}{\pi d} = \frac{P}{\pi d}$$

We now want to generalize this result to all convex polygons. Consider a convex polygon with sides $S_1, S_2, \ldots S_n$. A plate in the shape of this convex polygon will intersect a line at 0 or 2 points so

$$p = \sum_{i,j} P(S_i S_j)$$

Then, for each i, we will get

$$P(S_i) = \sum_{j \neq i} P(S_i S_j)$$

Summing over all i, we get

$$\sum_i P(S_i) = 2p$$

Let s_i be the length of S_i for all i. Then, from Theorem 1.2,

$$P(S_i) = \frac{2}{\pi}\frac{s_i}{d}$$

for all i From our previous equation, we get

$$p = \frac{\sum_i s_i}{\pi d} = \frac{P}{\pi d}$$

Therefore, our equation holds for plates in the shape of convex polygons.

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3. Buffon's Noodle Problem

Another extension of Buffon's Needle Problem is found by replacing the needle that is a straight line with a curved needle that can be thought of as a wet noodle. This extension is known as Buffon's Noodle Problem. The key difference between Buffon's Needle Problem and Buffon's Noodle Problem is that while a line needle can only cross a given line once, a wet noodle can cross the same line multiple times.

Theorem 3.1. *For a wet noodle, N, of length l thrown onto a ruled paper with lines at distance d apart, the expected number of crossings of N, given by $E(N)$ is*

$$E(N) = \frac{2l}{\pi d} \tag{3.1}$$

Proof. Consider a sequence of polygonal lines $L_1, L_2, \ldots$ that approximate N and segments L_i of the line denoted as $L_{i1}, L_{i2}, \ldots$ with $L_i = L_{i1} + L_{i2} + \cdots + L_{in}$. Let l_{ij} be the length of L_{ij}. For a sufficiently large N, we will have $l_{ij} < d$ so $E(L_{ij})$ is the probability of L_{ij} producing a crossing from Theorem 1.2. Therefore,

$$E(L_{ij}) = \frac{2l_{ij}}{\pi d}$$

Now, let l_i be the length of L_i. As i approaches infinity, l_i will approach l. If, for any i, $e(L_i) = e(L_{i1}) + ... + e(L_{in})$, then

$$E(L_i) = \sum_j \frac{2l_{ij}}{\pi d} = \frac{2}{\pi d} \sum_j l_{ij} = \frac{2l_i}{\pi d}$$

Therefore, as i approaches infinity, we will get

$$E(N) = \frac{2l}{\pi d}$$

and we will be done. Therefore, to prove Theorem 3.1, we need to show that $E(L_i) = E(L_{i1}) + \cdots + E(L_{i2})$. If we can prove this identity for $n = 2$, then we can use induction to extend it to all $n \geq 2$. Therefore, it suffices to show that for segments L and L',

$$E(L + L') = E(L) + E(L')$$

To prove this identity, we can look at cases based on the number of crossings. Let A denote the event that only L crosses a line, B denote the event that only L' crosses a line, and C denote the event that L and L' both cross a line. Events A and B will result in one crossing while C will result in two crossings. Therefore,

$$E(L + L') = E(A) + E(B) + 2E(C)$$

Additionally, looking at the events in which L crosses a line and the events in which L' cross a line, we get

$$E(L) = E(A) + E(C)$$

$$E(L') + E(B) + E(C)$$

It follows that

$$E(L + L') = E(L) + E(L')$$

Therefore, from our previous results,

$$E(N) = \frac{2l}{\pi d}$$

This result shows that the expected number of crossings for a needle depends only on the length of the needle and is not affected by the shape of the needle. An interesting application of this idea is Barbier's Theorem.

Theorem 3.2 (Barbier's Theorem). *Any closed curve of constant with d will have perimeter*

$$P = \pi d$$

A circle of diameter d will have an expected number of crossings of 2 and circumference $d\pi$. In general, any closed curve of diameter d will have an expected number of crossings of 2. Therefore, since the expected value of the number of crossings only depends on the perimeter of the curve, the curve will have perimeter equal to the circumference of the circle. Therefore, the perimeter of the closed curve will be $P = \pi d$.

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4. Buffon's Ball Problem

Buffon's Needle Problem can also be generalized to three dimensions in a problem known as Buffon's Ball Problem. Instead of a two-dimensional needle, Buffon's ball problem observes the probability of a needle contained inside a sphere lying above one of the lines on the paper.

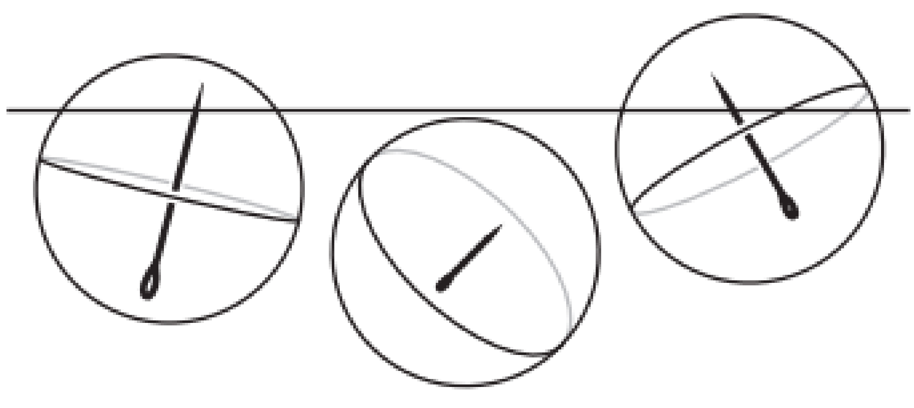

Figure 3. Examples of needles contained in a sphere, Image from [Ric06]

Theorem 4.1. *For a needle of length l contained in a ball of diameter l, the probability that the needle lies over one of the lines of a lined paper with lines at length $d \geq l$ apart is*

$$p = \frac{l}{2d} \tag{4.1}$$

Proof. Let x be the distance between the base of the ball and the closest line. Then, the probability that the needle crosses this line is

$$P(x) = \frac{A(x)}{\pi l^2}$$

where $A(x)$ is the area of the region of the sphere that creates a crossing and πl^2 is the area of the sphere. No crossing can be created for $x > \frac{l}{2}$ so $P(x) = 0$ when $x > \frac{l}{2}$. When $x \leq \frac{l}{2}$, there are two diametrically opposite regions on the top and bottom for which the tip of the needle can be placed which are shown below.

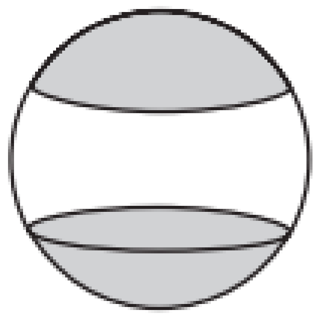

Figure 4. Image from [Ric06]

From a result by Archimedes, the area of the region is $2\pi rh$ where r is the radius of the sphere and h is the height of the region. Plugging in our values of $r = \frac{l}{2}$ and $h = \frac{l}{2} - x$, with x being the distance from the center of the sphere to the given region, we get

$$A(x) = l^2\pi\left(1 - \frac{2x}{l}\right)$$

This gives us

$$P(x) = 1 - \frac{2x}{l}$$

We now integrate over all possible distances x to get

$$p = \frac{2}{d}\int_0^{\frac{d}{2}} P(x)dx = \frac{2}{d}\int_0^{\frac{L}{2}} \left(1 - \frac{2x}{l}\right) dy$$

We can evaluate the given integral by finding the area of a triangle of height 1 and width $\frac{l}{2}$. Therefore,

$$p = \frac{l}{2d}$$

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Unlike the previous two problems, this extension to Buffon's Needle Problem gives us a probability formula that does not involve π. However, This extension can build on Buffon's Noodle Problem, with a curved needle of length l being placed inside a ball having an expected number of crossings of

$$E = \frac{l}{2d}$$

This also tells us that a curve of length l is expected to intersect parralel planes at distance d apart, an expected number of times of $E = \frac{l}{2d}$

5. The Buffon-Laplace Needle Problem

Buffon's Needle Problem can also be extended to a two dimensional board. This extension is known as the Buffon-Laplace Needle Problem in which a needle is dropped onto a board divided into rectangles of equal width and height.

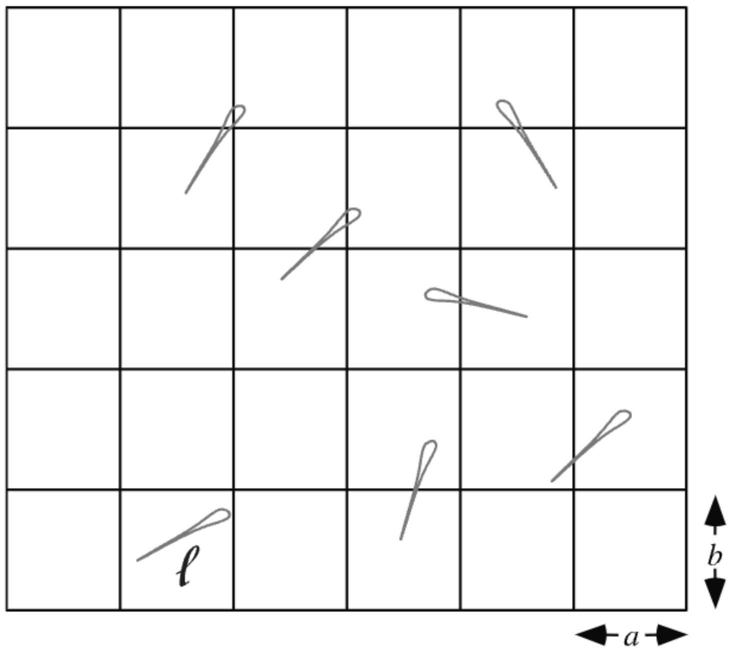

Figure 5. Board for Buffon-Laplace Needle Problem (Image from Wolfram Math World)

Theorem 5.1. *Consider a board lined with vertical lines a distance a apart from each other and horizontal lines a distance b apart from each other and a needle of length l such that $l < a$ and $l < b$. The probability that the needle intersects one of the boundaries of the rectangles on the board is*

$$p = \frac{2l(a+b) - l^2}{\pi ab} \tag{5.1}$$

Proof. To find the probability that the needle intersects the boundary of a rectangle, we first want to find the complementary probability, or the probability that the needle is entirely contained in a rectangle. Consider one of the rectangles $ABCD$ such that $AB = a$ and $BC = b$. We can place this rectangle on the coordinate plane with A at the origin, AB on the x-axis, and AD on the y-axis. The position of the needle can be represented by variables x, y, and θ with $0 < x < a$, $0 < y < b$, and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ where x and y are the coordinates of the middle point of the needle and θ is the angle the needle makes to the horizontal. The domain will then form a parralelepiped.

To find the probability that the needle will be enclosed inside ABCD, we consider the parts of the domain for fixed θ. We can take the projection of this region onto the xy-plane to get a rectangle $PQRS$ contained inside $ABCD$. From a result by Upsensky, we get

$$F(\theta) = [PQRS] = ab - bl\cos\theta - la|\sin\theta| + \frac{1}{2}l^2|\sin 2\theta|$$

To find the volume of the region of the domain for which the corresponding needle will lie completely inside $ABCD$, we integrate over all possible θ to get

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\theta)d\theta = \pi ab - 2bl - 2al + l^2$$

Therefore, since the volume of the domain is πab, the probability that a needle lands on the boundary of a rectangle is given by the complement of the probability that a needle lands inside a rectangle and is equal to

$$p = \frac{2l(a+b) - l^2}{\pi ab}$$

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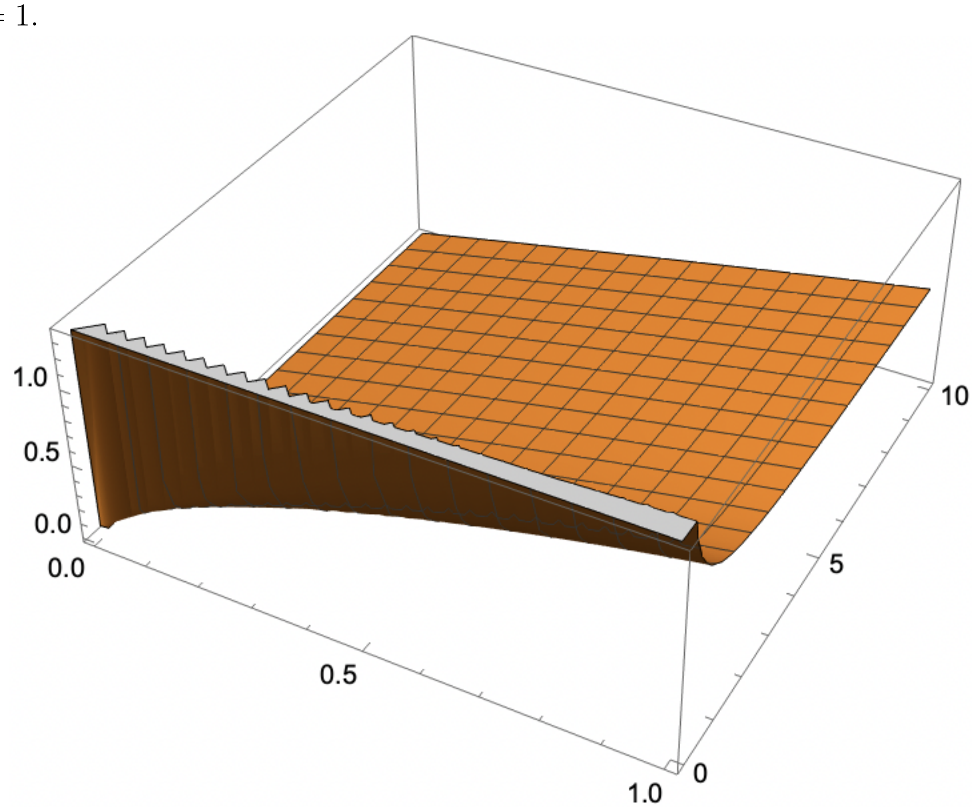
This also gives us another way of approaching Buffon's Needle Problem in one dimension by considering $a = \infty$. Then, we get

$$\lim_{a\to\infty} p = \lim_{a\to\infty} \left(\frac{2l}{\pi b} + \frac{2l}{\pi a} - \frac{l^2}{\pi ab} \right) = \frac{2}{\pi}\frac{l}{b}$$

Using Mathematica, we get the following probability plot for $0 < l < 1$, $0 < a < 10$ and fixed $b = 1$.



6. Experiments with Buffon's Needle Problem

Buffon's Needle Problem is famous for allowing one to experimentally determine the value of π in a simple fashion. However, this method is poor in practice,as it takes many trials to obtain a good approximation for π. A result by O'Gorman showed that it takes approximately 10^{2n+2} trials to obtain n digits of accuracy. Therefore, if one was able to toss a needle every second and record its outcome, it would take 11.57 days to attain two decimal places of accuracy, which is very inefficient.

A close approximation of π using this method was Lazzarini's approximation who obtained 1808 crossings out of 3048 trials to obtain $\pi \approx 3.1415929$. However, this result can be attributed to extreme luck, as a difference of one crossing could significantly change the approximation or data manipulation, with 1808 being a multiple of 13, $\frac{5}{6}(3048)$ being a multiple of 355 and $\pi \approx \frac{355}{113}$ being a well-known approximation of π. To be 95% confident of achieving a similar approximation, one would need to drop approximately 134 trillion needles. As a result, questions have arisen over whether or not Lazzarini truly performed the experiment.

Other experiments have been performed as well to approximate π to a difference of less than 0.02. Experiments performed by astronomor R. Wolf with a needle of length 36 mm and lines at distance 45 mm apart yielded 2532 crossings for 5000 trials. This led to the approximation $\pi \approx 3.1596$ which is close to the real value $\pi = 3.1415\ldots$ Another experiment performed by Ambrose Smith had 1213 crossings for 3204 trials and a needle of length equal to $\frac{3}{5}$ the distance between lines. This gave another close approximation with $\pi = 3.1412$.

[Mar14] [Ric06] [Ups37] [Wei22b] [Wei22a] [OGo22] [Ram69]

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