# SCISSORS CONGRUENCE

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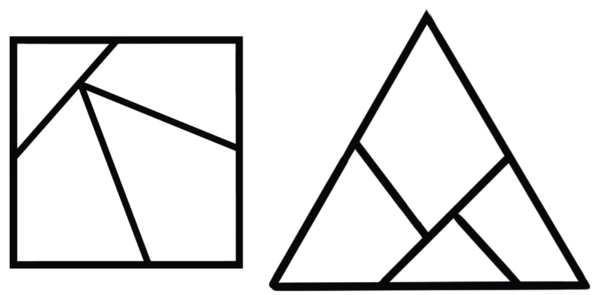
## 1. Introduction

A polygon is defined as a figure on a plane, closed with at minimum three straight sides. A simple polygon does not intersect itself nor have any holes. When we refer to a polygon as we continue, we shall be referring to a simple polygon specifically. If we cut up certain polygons using straight lines, we can rearrange the smaller polygons these cuts make into another, different polygon. We shall explore this idea further in equivalence relations in multiple dimensions and different geometries.

## 2. Scissors Congruence in 2D

**Definition 2.1.** If polygons $P$ and $Q$ are decomposed into sets of polygons $P_1, P_2, ..., P_n$ and $Q_1, Q_2, ...Q_n$, and $P_i$ and $Q_i$ are congruent for $1 \leq i \leq n$, then they are *scissors congruent*.



In the figure above, the square on the left can be cut up into four polygons that can be rearranged to form the triangle on the right. Thus, the square and triangle are scissors congruent. This particular example is called a Dudeny dissection, and is a type of hinge dissection.

**Definition 2.2.** The *degree of decomposability* is the minimum number of pieces the first polygon needs to be broken into in order to reform as the second polygon. If we call our initial polygon $P$ and our final polygon $Q$, we can write the degree of decomposability with notation: $\sigma(A, B)$.

It is pretty clear that since polygons $P$ and $Q$ are scissors congruent, they have the same area, as they are made up of the same smaller shapes. However, what about the converse?

**Theorem 2.3** (Wallace-Bolyai-Gerwein Theorem). *Any two polygons of equal area can always be split into a finite number of polygons that are pairwise congruent.*

*Proof.* We think of this theorem in terms of equidecomposability instead.

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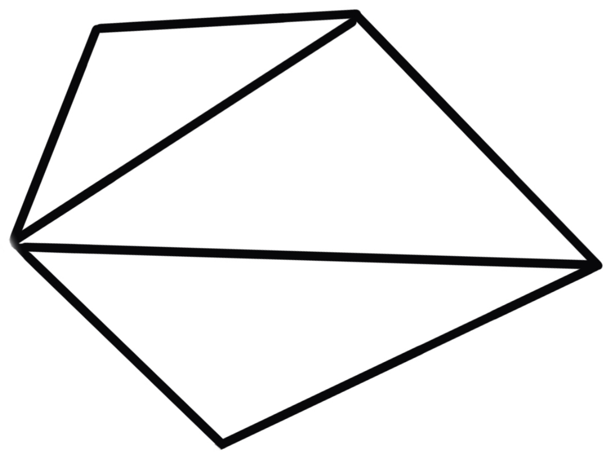
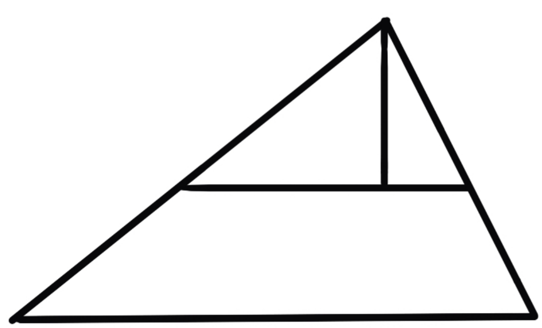
**Definition 2.4.** Two polygons are *equidecomposable* if they can be split into a finite number of congruent triangles.

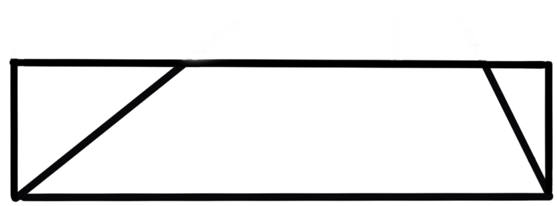
**Lemma 2.5.** *Every simple polygon can be triangulated. If a polygon has $n$ sides, it can be split into $n-2$ triangles.*



Each triangle can then be divided into a trapezoid and two right triangles with equal heights.



The two triangles can then be reorientated on the trapezoid to form a rectangle.



We do this with every triangle that makes up the polygon, and are left with $n-2$ rectangles. These rectangles can be decomposed and reformed rather easily to make different rectangles, each with the same width. Combining all these rectangles, we form one big rectangle.

We repeat this with the second polygon and are left with another rectangle. As these two rectangles have the same area, they can be decomposed and reformed to form each other. Thus, any two polygons of equal area are always scissors congruent.

An interactive demonstration of this can be found at: https://dmsm.github.io/scissors-congruence/. ■

## 3. The 3D

**Theorem 3.1** (Hilbert's Third Problem). *Two polyhedra of equal volume cannot always be cut into a finite number of polyhedra to make the other.*

This was a question posed by David Hilbert in his list of twenty-three mathematics problems. It was the first to be solved, in the same year itself, by his student Max Dehn. Dehn confirmed this conjecture by providing an example in which two polyhedra with equal volume could not be cut to make the other.

With scissors congruence in the second dimension, the only invariant was area. In the third dimension, we have volume and the Dehn invarient, as shown by Sydler in 1965.

**Definition 3.2.** We denote the Dehn invarient as $D(P)$. If polyhedra $P$ is divided into $k$ different polyhedron, it can be expressed as following:

$$D(P) = D(P_1) + D(P_2) + ... + D(P_k)$$

Every cube has a Dehn invarient of 0.

*Proof.* Similar to before, we try to think of it

We begin with Bricard's Condition.

**Theorem 3.3** (Bricard's Condition). *Given two equidecomposable polyhedra $P$ and $Q$, their dihedral angles $\alpha_1, ..., \alpha_r$ and $\beta_1, ..., \beta_s$ satisfy*

$$m_1\alpha_1 + ... + m_r\alpha_r = n_1\beta_1 + ... + n_s\beta_s + k\pi \tag{3.1}$$

*for some integer $k$ and positive integers $m_1, ..., m_r$ and $n_1, ..., n_s$.*

Dehn used the following example to prove a counterexample. We choose a regular tetrahedron, which has dihedral angles of $arccos\frac{1}{3}$. We choose a cube, which has dihedral angles of $\frac{\pi}{2}$. We can use Bricard's Condition to express this as:

$$m_1 arccos\frac{1}{3} = n_1\frac{\pi}{2} + k\pi$$

Solving for $k$, we get $k = m_1\frac{1}{\pi}arccos\frac{1}{3} - \frac{1}{2}n_1$. However the right hand side is irrational, which contradicts with $k$, which is an integer. Thus, the tetrahedron is not equidecomposable to the cube although they have the same volume. ■

## 4. Further

We could attempt to look further into higher dimensions and across different geometries. So far, these relations have been found in the second dimension of hyperbolic and spherical geometries. However, it is yet to be solved for other dimensions or geometries.

## References

[1] Abhijit Champanerkar, *Scissors Congruence & Hilbert's 3rd Problem*, Feb. 2020.
[2] Maeve Coates Welsh, *Scissors Congruence*, Sept. 2016.
[3] Inna Zakharevich, "Perspectives on Scissors Congruence", *Bulletin of the New American Mathematical Society*, Vol. 53, pp. 269-294, Apr. 2016.