Monsky's Theorem

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1 Introduction

Monsky's Theorem is a deceptively simple theorem in geometry whose only proof involves, surprisingly, combinatorial arguments and 2-adic numbers.

Theorem 1.1 (Monsky's Theorem)**.** *It is impossible to dissect a square into an odd number of triangles with equal areas.*

We then provide a generalization to the regular n-gon.

Theorem 1.2 (Generalization of Monsky's Theorem)**.** *For $n \geq 5$, it is only possible to equidissect a regular n-gon into m triangles when m is a multiple of n.*

2 p-adic

2.1 Valuations

Definition 2.1. A real valuation on a field K is a function $v(x)$ from K into $\mathbb{R} \cup \{\infty\}$ such that, for all $a, b \in K$,

1.
2. $v(x) \leq \infty$ with equality if and only if $x = \infty$
3. $v(xy) = v(x) + v(y)$
4. $v(x+y) \geq \min\{v(x), v(y)\}$ with equality if $v(x) \neq v(y)$ (Ultrametric Inequality)

Proposition 2.2. *For any valuation $v(x)$, we have $v(1) = 0$, $v(-x) = v(x)$, and $v(x^{-1}) = -v(x)$.*

Proof. Easily, $v(1) = v(1) + v(1)$, so $v(1) = 0$. Then, $v(-1) + v(-1) = v(1) = 0$, sso $v(-1) = 0$ as well. $v(-x) = v(-1) + v(x) = v(x)$, and $v(x^{-1}) + v(x) = v(1) = 0$, so $v(x^{-1}) = -v(x)$, as claimed. $\square$

We now consider the p-adic valuation, denoted $v_p(x)$ for a prime p.

Definition 2.3. Consider any prime p. Any rational nonzero $r \in \mathbb{Q}$ can be uniquely expressed as $p^k \frac{a}{b}$ such that k is an integer and a, b are relatively prime to p. Then, let the p-adic value of r be

$$v_p(r) = k, \qquad v_p(0) = \infty$$

Example. For the 2-adic valuation, $v_2(15) = 0$, $v_2(30) = 1$, $v_2(-64) = 6$, and $v_2\left(\frac{15}{12}\right) = -2$.

In general, for integers n, $v_p(n)$ is exactly the number of factors of p in n.

Theorem 2.4. *$v_p(x)$ is a real valuation.*

Proof. Condition 1 of Def. 2.1 is satisfied easily. Furthermore we can verify condition 2: we let $v_p(x) = m, v_p(y) = n$, so $x = p^m \frac{a}{b}$ and $y = p^n \frac{c}{d}$. Then, $xy = p^{m+n}\left(\frac{ac}{bd}\right)$, and so $v_p(xy) = m + n = v_p(x) + v_p(y)$, as desired.

Finally, we can check the ultrametric inequality. Without loss of generailty, assume $v_p(x) = m$ and $v_p(y) = n$ with $m \geq n$. Again let $x = p^m \frac{a}{b}$ and $y = p^n \frac{c}{d}$. Then,

$$x + y = p^n \left(\frac{p^{m-n}ad + bc}{bd}\right)$$

The fraction's denominator is relatively prime to p, so we have that $v_p(x+y) \geq n = v_p(y)$. Thus, in general, we have that $|x+y|_p \geq \max\{v_p(x), v_p(y)\}$. Moreover, if $m > n$, then the numerator is also relatively prime to p, as p^{m-n} is divisible by p whereas bc is not. Thus, we have that equality holds if $v_p(x) \neq v_p(y)$, as desired. □

However, in the proof of Monsky's theorem, it is actually more convenient to use not the p-adic valuation but the p-adic absolute value.

2.2 Absolute Values

Definition 2.5. An absolute value on a field K is a function $|x|$ from K into $\mathbb{R}^{\geq 0}$ such that, for all $a, b \in K$,

1. $|x| = 0$ iff $x = 0$
2. $|xy| = |x|\,|y|$
3. $|x+y| \leq |x| + |y|$ (Triangle Inequality)

Proposition 2.6. *For any absolute value $|x|$, we have $|1| = 1$, $|-x| = |x|$, and $|x^{-1}| = |x|^{-1}$.*

Proof. Easily, $|1| = |1 \cdot 1| = |1|\,|1| \neq 0$. Thus, $|1| = 1$. We then have $1 = |1| = |-1|\,|-1|$, so we must have $|-1| = 1$. As a result, $|-x| = |-1|\,|x| = |x|$. Finally, $|x|\,|x^{-1}| = |1| = 1$, so $|x^{-1}| = |x|^{-1}$, as claimed. □

The standard absolute value function on the reals is an absolute value on $\mathbb{R}$. However, there turns out to be many other absolute values as well.

Definition 2.7. Consider any prime p. For any rational nonzero $r \in \mathbb{Q}$, define

$$|r|_p = p^{-v_p(r)}, \qquad |0|_p = 0$$

Example. For the 2-adic absolute value, $|15|_2 = 1$, $|30|_2 = \frac{1}{2}$, $|-64|_2 = \frac{1}{64}$, and $\left|\frac{15}{12}\right|_2 = 4$.

Theorem 2.8. *For any prime p, $|x|_p$ is an absolute value on $\mathbb{Q}$. Moreover, it is non-archimedian, meaning that*

$$|x+y|_p \leq \max\{|x|_p, |y|_p\} \text{ with equality if } x \neq y$$

Proof. This proof follows directly from the properties of $v_p(x)$. □

2.3 Extensions to the Reals

We have only defined $v_p(x)$ and $|x|_p$ on the rationals. However, it turns out we can also extend any real valuation to $\mathbb{R}$.

Theorem 2.9 (Chevalley's Theorem)**.** *Any real valuation $v(x)$ on a field K can be extended to a real valuation $v(x)'$ any field L containing K, such that if $x \in L$, $v(x)' = v(x)$.*

In this paper, we unfortunately accept this theorem without proof. The proof can instead be found in [4]. It immediately implies that there also exists an absolute value $|x|_p$ on the reals, which we shall use in the proof of Monsky's theorem.

3 Monsky's Theorem

3.1 A Coloring of the Plane

We now start to prove Monsky's Theorem. Consider the square with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 0)$. Now, color every point inside the square according to the following rules:

$$(x, y) \text{ is colored } \begin{cases} \text{blue} & \text{if } |x|_2 \geq |y|_2, |x|_2 \geq 1 \\ \text{green} & \text{if } |x|_2 < |y|_2, |y|_2 \geq 1 \\ \text{red} & \text{if } |x|_2, |y|_2 < 1 \end{cases}$$

This coloring has a couple important properties.

Theorem 3.1. *The coloring is translation-invariant with respect to red points. That is, if $(x_1, y_1), (x_2, y_2)$ are red points, then $(x_1 + x, y_1 + y)$ and $(x_2 + x, y_2 + y)$ have the same color for all x, y.*

Proof. Note the origin is colored red. We can show that for any red point (x_r, y_r), (x, y) and $(x - x_r, y - y_r)$ have the same color for all x, y, which will then imply the theorem.

First, assume (x, y) is red, so that $|x|_2, |y|_2, |x_r|_2, |y_r|_2 < 1$. Then,

$$|x - x_r|_2 \leq \max\{|x|_2, |x_r|_2\} < 1$$

A similar statement holds for $|y - y_r|_2 < 1$. As a result, this means that $(x - x_r, y - y_r)$ must be colored red.

Next, let (x, y) be green, with $|x|_2 < |y|_2 \geq 1$. Since $|y_r|_2 < 1$, $|y|_2 > |y_r|_2$.

$$|y - y_r|_2 = \max\{|y|_2, |y_r|_2\} = |y|_2 > 1$$

Note that $|x|_2 < |y|_2$, and that $|x_r|_2 < 1 \leq |y|_2$. Thus,

$$|x - x_r|_2 < |y|_2 = |y - y_r|_2$$

Thus, $(x - x_r, y - y_r)$ must be colored green.

The last case for when (x, y) is blue is analagous to the green case, and we are done. □

Definition 3.2. Let a rainbow triangle in the plane be a triangle whose vertices are red, blue, and green, in any order.

Theorem 3.3. *Any rainbow triangle cannot have area $\frac{1}{n}$ for odd n. Also, any line in the plane contains at most 2 colors.*

Proof. Let the red vertex be (x_r, y_r). Translate the triangle such that (x_r, y_r) lands on the origin, which is a red point. Let the point that the green vertex lands on be (x_g, y_g), which from before we know is green. Do the same for the blue vertex, which lands on (x_b, y_b).

We note the area of this triangle remains unchanged under translation, and is

$$\frac{1}{2}\begin{vmatrix} x_g & x_b \\ y_g & y_b \end{vmatrix} = \frac{1}{2}(x_g y_b - x_b y_g)$$

From the colors, we know $|x_b|_2 \geq |y_b|_2$ and $|y_g|_2 > |x_g|_2$. Multiplying the two together yields $|x_b y_g|_2 > |x_g y_b|_2$. Note that this necessarily means that the area of this triangle is non-zero, so the red, blue, and green points cannot lie on a line. Thus, any line in the plane must contain at most 2 colors.

Now we compute the 2-adic absolute value of the area of the triangle.

$$\left|\frac{1}{2}(x_g y_b - x_b y_g)\right|_2 = \left|\frac{1}{2}\right|_2 |x_g y_b - x_b y_g|_2 = 2\,|x_g y_b - x_b y_g|_2$$

$$2\,|x_g y_b - x_b y_g|_2 = 2\max\{|x_g y_b|_2\,, |x_b y_g|_2\} = 2\,|x_b y_g|_2 = 2\,|x_b|_2\,|y_g|_2$$

From the colors, we know $|x_b|_2 \geq 1$ and $|y_g|_2 \geq 1$.

$$2\,|x_b|_2\,|y_g|_2 \geq 2$$

As a result, if the area of the rainbow triangle is A, $|A|_2 > 1$. However, this means A cannot be $\frac{1}{n}$, where n is odd, because then $|A|_2 = 1$, as claimed. □

3.2 Sperner's Lemma

Call an edge purple if one of its endpoints is red and one is blue.

Lemma 3.4. *In any triangulation, the square S contains an odd number of purple edges on its boundary.*

Proof. The point $(0, 0)$ is red, the points $(1, 0)$ and $(1, 1)$ are blue, and the point $(0, 1)$ is green.

Consider all the vertices and edges of the triangulation. Note that the vertices of the square must be vertices in the triangulation.

Now consider the edge of S from $(0, 0)$ to $(1, 0)$, which must only contain red and blue points. Some vertices of the triangulation may lie on this segment, but note that in total, there are an odd number of purple edges. This is because, starting at $(0, 0)$ and moving along successive vertices to $(1, 0)$, you start at a red point but end at a blue point. Thus, you must

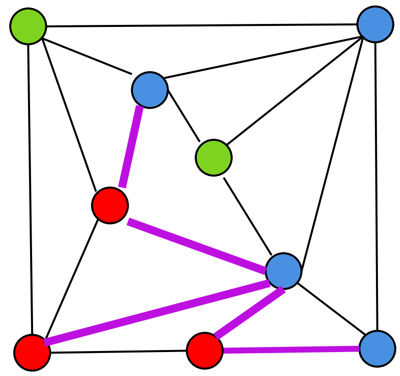

Figure 1: A dissection of a square, with purple edges colored purple.

have switched colors an odd number of times, which means there are an odd number of purple edges.

None of the other edges of the square S can contain an odd number of purple edges, since none of them contain both red and blue points. Note on the edge from $(1, 0)$ to $(1, 1)$, there could be purple edges, but since it starts and ends at blue, the number of purple edges must be even.

In total, there must be an odd number of purple edges on the boundary of S. □

Now, we can prove that there must exist a rainbow triangle in S via Sperner's Lemma:

Theorem 3.5 (Sperner's Lemma)**.** *Assume a polygon P has an odd number of purple edges on its boundary. Every dissection of P into triangles contains an odd number of rainbow triangles.*

Proof. Consider each triangle in the interior of P. If it is not a rainbow triangle, then note that it contains an even number of edges that have a red point and a blue point as endpoints. For each of these edges, a similar argument from before shows that there will be an odd number of purple segments that lie on the edge. However, because there are an even number of such edges, there are also an even number of purple edges on this triangle's boundary.

However, if the triangle is rainbow, then it only has 1 edge that has a red point and a blue point as endpoints. Then, this rainbow triangle must contain an *odd* number of purple segments on its boundary.

Sum over all triangles the number of purple edges on its boundary. Note that this sum will count each purple edge in the interior of P twice but the purple edges on the boundary of P only once, so this sum must be odd. However, since each non-rainbow triangle contributes an even number to this sum, there must be an odd number of rainbow triangles. Specifically, there must exist a rainbow triangle! □

Finally, we prove Monsky's Theorem:

Theorem 3.6 (Monsky's Theorem)**.** *It is impossible to dissect a square into an odd number of triangles with equal areas.*

Proof. There must exist an odd number of rainbow triangles in any equidissection of the square. However, if n is odd, then this rainbow triangle cannot have area $\frac{1}{n}$, and we are done. ☐

4 Generalization to a Regular n-gon

It is worth wondering whether Monsky's theorem generalizes in some way to other regular n-gons, and in fact, it does.

Generally, for any polygon P, the spectrum of P is the set of all $m \in \mathbb{N}$ for which P can be equidissected into m triangles with equal areas. Note that if m is in the spectrum of P, nm is too, since it is possible to equidissect all the triangles into n smaller triangles.

Due to this, it is often the case that the spectrum of a polygon is the set of all multiples of m, for some m. In this case, we call the polygon *principal* and we write its spectrum as $\langle m \rangle$. Triangles are principal with spectrum $\langle 1 \rangle$, and squares we have proven are principal with spectrum $\langle 2 \rangle$.

Now, we aim to prove the following theorem:

Theorem 4.1 (Generalization of Monsky's Theorem)**.** *A regular n-gon with $n \geq 5$ has spectrum $\langle n \rangle$. That is, it is only possible to dissect the n-gon into m triangles when m is a multiple of n.*

4.1 p-adic Valuation

Previously, we have used the p-adic absolute value. However, we shall switch here to use the p-adic valuation, which is defined again below for convenience:

Definition 4.2. Consider any prime p. Any rational nonzero $r \in \mathbb{Q}$ can be uniquely expressed as $p^k \frac{a}{b}$ such that k is an integer and a, b are relatively prime to p. Then, let the p-adic value of r be

$$v_p(r) = k, \qquad v_p(0) = \infty$$

Recall the following properties, which we have proven before:

1.
2. $v_p(x) \leq \infty$ with equality if and only if $x = \infty$
3. $v_p(xy) = v_p(x) + v_p(y)$
4. $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$ with equality if $v_p(x) \neq v_p(y)$ (Ultrametric Inequality)
5. $v_p(1) = 0$, $v_p(-x) = v_p(x)$, and $v_p(x^{-1}) = -v(x)$.

By Chevalley's Theorem, there exists an extension of $v_p(x)$ into any field extension of $\mathbb{Q}$. Here, we will be using an extension of $v_p(x)$ into the complex numbers $\mathbb{C}$, which we assume exists.

4.2 Colorings: Reprise

Let p be a prime that divides n. we will show that if a regular n-gon can be equidissected into m triangles, then $v_p(m) \geq v_p(n)$, and since m and n are both integers, this shows that the number of factors of p in m is at least the number in n. If this holds for all primes p, this will show n divides m.

Let us once again color points in the plane according to the same rules from Monsky's theorem, just with a general prime p:

$$(x, y) \text{ is colored } \begin{cases} \text{blue} & \text{if } v_p(x) \leq v_p(y), v_p(x) \leq 0 \\ \text{green} & \text{if } v_p(x) > v_p(y), v_p(y) \leq 0 \\ \text{red} & \text{if } v_p(x), v_p(y) > 0 \end{cases}$$

There's an important lemma we must consider:

Lemma 4.3. *Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane. If $A+B = (x_1+x_2, y_1+y_2)$ is colored red, A and B must be the same color.*

Proof. We recall that the coloring is translation-invariant, whose proof is exactly the same as Theorem 3.1. Translate the point A by $-(A+B)$. Because both the origin and $A+B$ are red, this new point $A-(A+B) = -B$ has the same color as A. However, B and $-B$ must have the same color because $v_p(-x) = v_p(x)$, and we are done. □

Theorem 4.4. *For any rainbow triangle R with area $[R]$, we have that*

$$v_p(2 \cdot [R]) \leq 0$$

Also, any line in the plane contains at most 2 colors.

Proof. This coloring is translation-invariant, so translate the triangle such that the red vertex lies on the origin. The green and blue vertices then become (x_g, y_g), (x_b, y_b), respectively.

We note that the area of the triangle, $[R]$ is once again

$$[R] = \frac{1}{2}(x_g y_b - x_b y_g) \qquad \Rightarrow \qquad 2[R] = x_g y_b - x_b y_g$$

From the colors, we know $v_p(x_b) \leq v_p(y_b)$ and $v_p(y_g) < v_p(x_g)$. Adding the two and using the properties of $v_p(x)$ yields $v_p(x_b y_g) < v_p(x_g y_b)$. Because the inequality is strict, $x_b y_g \neq x_g y_b$ and the area of the triangle is nonzero. Therefore, any line in the plane cannot contain all 3 colors, as before.

Now, using this inequality we have

$$v_p(2[R]) = v_p(x_g y_b - x_b y_g) = \min\{v_p(x_g y_b), v_p(x_b y_g)\} = v_p(x_b y_g)$$

$$= v_p(x_b) + v_p(y_g) \leq 0$$

The last inequality follows due to $v_p(x_b), v_p(y_g) \leq 0$ from their colors. □

Now, say there exists a rainbow triangle in an equidissection of some polygon P into m triangles.

The area of the rainbow triangle R must be $\frac{1}{m} \cdot [P]$, so we have

$$v_p\left(\frac{2}{m} \cdot [P]\right) = v_p(2 \cdot [R]) \leq 0$$

As one of our properties of the valuation, $v_p\left(\frac{1}{m}\right) = -v_p(m)$, so

$$v_p(2 \cdot [P]) \le v_p(m)$$

We will use this inequality later, as a means to show our desired result that $v_p(m) \ge v_p(n)$.

4.3 Affine Transformations

Without loss of generality, let P be a regular n-gon inscribed in a circle with radius 1. Then, the area of P, which we denote $[P]$, is simply $[P] = \frac{1}{2}n \cdot \sin\frac{2\pi}{n}$ by standard geometrical techniques.

Like the proof for Monsky's theorem, we wish to show in any triangulation of P into m triangles, at least one triangle cannot have area $\frac{[P]}{m}$ when m is not a multiple of n. However, instead of directly using P, let us first apply an affine transformation so that it is easier to work with.

Let $\theta = \frac{2\pi}{n}$, and let P be the regular n-gon with vertices at $((\cos j\theta) - 1, \sin j\theta)$. Then, let T be the affine transformation described by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = -2\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Applying T to each the vertices of P yields

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = -2\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{bmatrix}\begin{bmatrix} (\cos j\theta) - 1 \\ \sin j\theta \end{bmatrix} = -2\begin{bmatrix} \cos\frac{\theta}{2}\cos j\theta - \sin\frac{\theta}{2}\sin j\theta - \cos\frac{\theta}{2} \\ \cos\frac{\theta}{2}\cos j\theta + \sin\frac{\theta}{2}\sin j\theta - \cos\frac{\theta}{2} \end{bmatrix}$$

Simplying using the cosine addition formula yields

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = -2\begin{bmatrix} \cos\left(j\theta + \frac{\theta}{2}\right) - \cos\frac{\theta}{2} \\ \cos\left(j\theta - \frac{\theta}{2}\right) - \cos\frac{\theta}{2} \end{bmatrix}$$

Now, using the fact that $\cos A - \cos B = -2\left(\sin\frac{A+B}{2} - \sin\frac{A-B}{2}\right)$, we have that the vertices of P' are

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = 4\begin{bmatrix} \sin\frac{(j+1)\theta}{2}\sin\frac{j\theta}{2} \\ \sin\frac{(j-1)\theta}{2}\sin\frac{j\theta}{2} \end{bmatrix}$$

We have described the vertices of (x_j, y_j), and importantly, note that $(x_0, y_0) = (0, 0)$. Also, note $x_{n-1} = 0$ and $y_1 = 0$.

We can also find the area of P', which must be the area of P times the determinant of the transformation T.

$$[P'] = [P] \cdot \det\left(-2\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{bmatrix}\right) = [P] \cdot 8\sin\frac{\theta}{2}\cos\frac{\theta}{2} = [P] \cdot 4\sin\theta$$

Substituting $[P] = \frac{1}{2}n \cdot \sin\frac{2\pi}{n}$ from before, we have

$$[P'] = 2\sin^2\theta$$

4.4 Complex Numbers

Now, we wish to find the p-adic valuation of the area and the vertices of P', but these values all involve the sin function of some sort. To take full advantage of the multiplicativity of the p-adic valuation, we take a detour to the complex numbers.

Recall that we have extended $v_p(x)$ to the complex numbers, which is possible. If ω is any root of unity, i.e. $\omega^n = 1$ for some $n \in \mathbb{Z}$, then we have that $n \cdot v_p(\omega) = v_p(\omega^n) = v_p(1) = 0$, $v_p(\omega) = 0$.

Consider $\sin\frac{k\theta}{2}$, with $\theta = \frac{2\pi}{n}$. Also, let $\zeta = e^{i\theta}$.

$$\sin\frac{k\theta}{2} = \frac{e^{i\frac{k\theta}{2}} - e^{-i\frac{k\theta}{2}}}{2i} = \frac{1}{2} \cdot ie^{-i\frac{k\theta}{2}} \cdot (1 - e^{ik\theta}) = \frac{1}{2} \cdot ie^{-i\frac{k\theta}{2}} \cdot (1 - \zeta^k)$$

Note here that $ie^{-i\frac{k\theta}{2}}$ is a root of unity, so its valuation is 0. Taking the p-adic valuation, we then have

$$v_p\left(\sin\frac{k\theta}{2}\right) = v_p\left(\frac{1}{2}\right) + v_p\left(ie^{-i\frac{k\theta}{2}}\right) + v_p\left(1 - \zeta^k\right) = v_p\left(\frac{1}{2}\right) + v_p\left(1 - \zeta^k\right)$$

Recall that from before, the vertices of P' are

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = \begin{bmatrix} 4\sin\frac{(j+1)\theta}{2}\sin\frac{j\theta}{2} \\ 4\sin\frac{(j-1)\theta}{2}\sin\frac{j\theta}{2} \end{bmatrix}$$

Let us take the p-adic valuation of x_j and y_j using our formula for $v_p(\sin x)$:

$$\begin{aligned} v_p(x_j) &= v_p(4) + v_p\left(\frac{1}{2}(1 - \zeta^{j+1})\right) + v_p\left(\frac{1}{2}(1 - \zeta^j)\right) \\ &= v_p\left(1 - \zeta^{j+1}\right) + v_p\left(1 - \zeta^j\right) \end{aligned}$$

Similarly,

$$v_p(y_j) = v_p\left(1 - \zeta^{j-1}\right) + v_p\left(1 - \zeta^j\right)$$

Here, we must unfortunately assume without proof some results from the theory of cyclotomic fields, whose proofs can be found in [5].

Lemma 4.5. *If p is odd and $\zeta^j \neq 1$, then $v_p(1 - \zeta^j) \leq \frac{1}{p-1}$ with equality iff ζ^j is a p-th root of unity. Also, if $p = 2$ and $\zeta^j \neq \pm 1$, then $v_p(1 - \zeta^j) \leq \frac{1}{2}$ with equality iff $\zeta^j = \pm i$.*

Now, we can show the following:

Lemma 4.6. *If $n \geq 4$, then $v_p(y_j) < 1$ if either*

1. *p is odd and $j \notin \{0, 1\}$*
2. *$p = 2$ and $j \notin \{0, 1, \frac{n}{2}, \frac{n}{2} + 1\}$*

Proof. First, let p be odd with $j \notin \{0, 1\}$, so ζ^j and ζ^{j-1} are both not 1. Recall

$$v_p(y_j) = v_p\left(1 - \zeta^{j-1}\right) + v_p\left(1 - \zeta^j\right) \leq \frac{1}{p-1} + \frac{1}{p-1} \leq 1.$$

This follows by the previous lemma. Equality only holds if $p = 3$ and ζ^j and ζ^{j-1} are both 3-rd roots of unity. This means ζ must too be a 3-rd root of unity, but since $\zeta = e^{\frac{2\pi}{n}}$ from before, we know this means $n = 3$. However, we required $n \geq 4$, so equality never holds for p odd.

Now, let $p = 2$ with $j \notin \{0, 1, \frac{n}{2}, \frac{n}{2} + 1\}$. Then ζ^{j-1} and ζ^j are both not ± 1, and we have

$$v_p(y_j) = v_p\left(1 - \zeta^{j-1}\right) + v_p\left(1 - \zeta^j\right) \leq \frac{1}{2} + \frac{1}{2} = 1$$

Equality only holds when ζ^{j-1} and ζ^j are $\pm i$, but this is impossible, and we are done. □

4.5 One Last Transformation

Recall that P is a regular n-gon, and P' is P under the affine transformation T. We have so far described the vertices and area of P', but we must apply one last transformation.

If p is odd, let $k \notin \{0, 1\}$ be the index such that $v_p(y_k)$ is as large as possible. If $p = 2$, instead let $k \notin \{0, 1, \frac{n}{2}, \frac{n}{2} + 1\}$ such that $v_p(y_k)$ is maximized.

Now, let $P^\star$ be the polygon P' under the mapping

$$(x, y) \to \left(\frac{x}{x_1}, \frac{y}{y_k}\right)$$

Lemma 4.7. *We claim*

$$v_p(2 \cdot [P^\star]) > v_p(n) - 1$$

Proof. We know the area of P' is $2\sin^2\theta$, from before. The area of $P^\star$ is then $\frac{[P']}{x_1 y_k}$.

$$v_p(2[P^\star]) = v_p(2[P']) - v_p(x_1) - v_p(y_k)$$

Consider the first term. $v_p(2[P']) = v_p((2\sin\theta)^2) = 2 \cdot v_p(2\sin\theta)$. Recall from before that $v_p(2\sin\theta) = v_p(1 - \zeta^2)$, so we have $v_p(2[P']) = 2 \cdot v_p(1 - \zeta^2)$, and

$$v_p(2[P']) = 2 \cdot v_p\left(1 - \zeta^2\right) - v_p(x_1) - v_p(y_k)$$

Substuting in our value for $v_p(x_1)$ from before yields

$$\begin{aligned} v_p(2[P']) &= 2 \cdot v_p\left(1 - \zeta^2\right) - \left(v_p\left(1 - \zeta^2\right) + v_p(1 - \zeta)\right) - v_p(y_k) \\ &= v_p(n) + v_p(1 + \zeta) - v_p(y_k) \end{aligned}$$

By the non-archimedian property of $v_p(x)$, $v_p(1 + \zeta) \geq \min\{v_p(1), v_p(\zeta)\} = \min\{0, 0\} = 0$, so $v_p(2[P']) \geq v_p(n) - v_p(y_k)$, and with $v_p(y_k) > 1$ by the previous lemma, we have the desired result. □

Putting it all together, we know that if there exists a rainbow triangle in an equidissection of $P^\star$, then we must have that $v_p(2 \cdot [P^\star]) \leq v_p(m)$. But then, we have shown that $v_p(m) > v_p(n) - 1$, and because m, n are integers, $v_p(m) \geq v_p(n)$, as desired.

Thus, it suffices to show that there exists a rainbow triangle in $P^\star$, which we shall do.

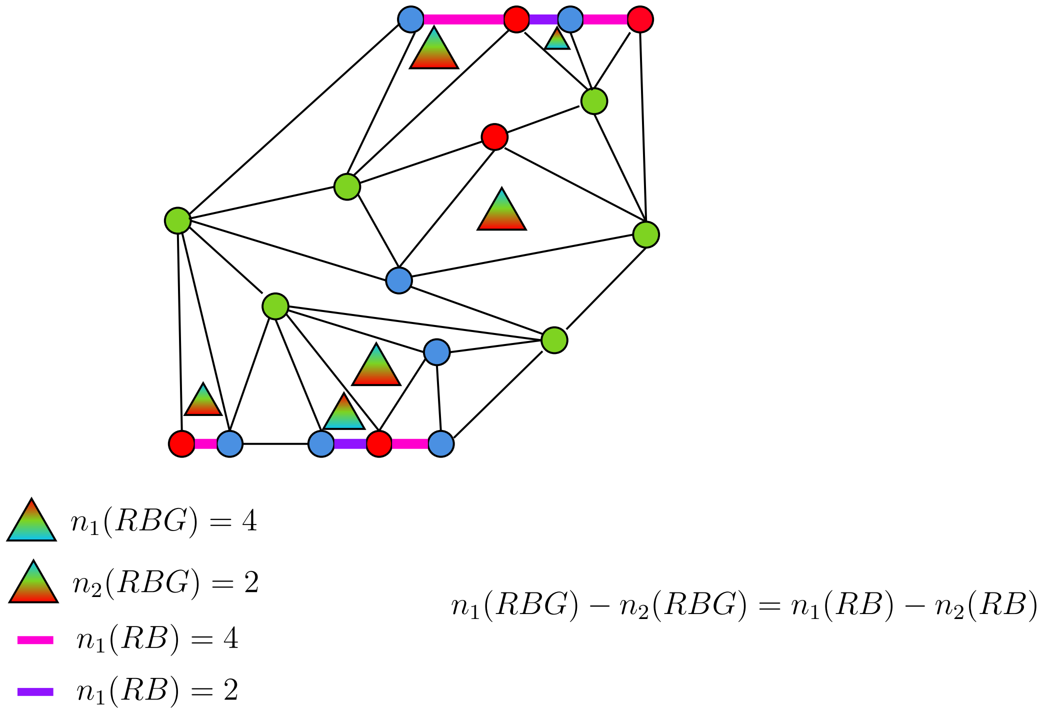

Figure 2: An illustration of Sperner's Lemma in Integral Form

4.6 Sperner's Lemma, Extended

For Monsky's Theorem, Sperner's Lemma sufficed. However, we will require the use of an extended version for a regular n-gon.

Theorem 4.8 (Sperner's Lemma, Integral Form)**.** *Consider a dissection of a polygon P where each vertex is either red, green, or blue, denoted R, G, and B respectively. Then, let $n_1(RBG)$ be the number of rainbow triangles whose vertices are colored red, blue, and green, in counterclockwise order, and let $n_2(RBG)$ be the number where it is in clockwise order. Let $n_1(RB)$ be the number of purple edges on the boundary of P that have the blue endpoint immediately counterclockwise of the red point, and let $n_2(RB)$ be the number that have the blue point immediately clockwise. Then,*

$$n_1(RBG) - n_2(RBG) = n_1(RB) - n_2(RB)$$

Notably, if $n_1(RB) \neq n_2(RB)$, there must be a rainbow triangle in P.

Proof. Consider the number of purple edges in the dissection that do not lie on the boundary of P. Sum over all triangle in the dissection the number of purple edges on its boundary such that the blue endpoint is immediately counterclockwise from the red endpoint minus the number of purple edges that have its blue endpoint immediately clockwise.

Note that this total sum must be $n_1(RB) - n_2(RB)$, for every purple edge in the interior is on the boundary of two triangles. For one triangle, it contributes 1 to the sum, while for the other, it contributes -1, for a total of 0. However, for edges on the boundary of P, note it is adjacent to only 1 triangle. It then contributes 1 to the sum if it is part of $n_1(RB)$ and -1 otherwise.

Meanwhile, in each triangle, if the triangle is not rainbow, then a little casework shows the sum for the triangle is necessarily 0. However, if it is a rainbow triangle that is part of $n_1(RBG)$, then it contributes 1, and -1 otherwise.

Thus, equating these terms yields

$$n_1(RBG) - n_2(RBG) = n_1(RB) - n_2(RB)$$

□

At long last, we have the tools necessary to show that there must be a rainbow triangle in $P^\star$.

Lemma 4.9. *In any equidissection of $P^\star$, there exists a rainbow triangle.*

Proof. If p is odd, consider the vertex $(\frac{x_0}{x_1}, \frac{y_0}{y_k}) = (0,0)$ of $P^\star$. This vertex is clearly colored red. Note that $(\frac{x_1}{x_1}, \frac{y_1}{y_k}) = (1,0)$ is immediately adjacent and is colored blue.

Now, note $(\frac{x_{n-1}}{x_1}, \frac{y_{n-1}}{y_k})$ has a x-coordinate of 0 and the valuation of its y-coordinate is $v_p(y_{n-1}) - v_p(y_k)$. By construction of k, this is nonpositive, and this point is colored green.

For all other points $(\frac{x_j}{x_1}, \frac{y_j}{y_k})$, we note $v_p(y_k) \geq v_p(y_j)$ so $v_p\left(\frac{y_j}{y_k}\right) \leq 0$. Therefore, these points cannot be red.

The only red vertex, then, is the vertex $(\frac{x_0}{x_1}, \frac{y_0}{y_k}) = (0,0)$, and it is adjacent to a blue and a green vertex. Thus, we note that there must be an odd number of red-blue edges on the boundary of $P^\star$. Here, we in fact only require the usual Sperner's Lemma, which tells us that there must be a rainbow triangle.

The case for when $p = 2$ is a little more complex. $P^\star$ is P under some affine transformation, so some qualities are still preserved. For example, $P^\star$ still has a center C, which by applying the transformations to the original center of P, which was $(-1,0)$ yields

$$C = \left(\frac{2\cos\frac{\theta}{2}}{x_1}, \frac{2\cos\frac{\theta}{2}}{y_k}\right).$$

Here, $x_1 = y_2$ by our formula for (x_j, y_j) from before, and so $x_1 = y_2 \leq y_k < 1$, also from before.

Note $2\cos\frac{\theta}{2} = e^{i\frac{\theta}{2}} + e^{-\frac{\theta}{2}} = e^{i\frac{\theta}{2}} \cdot (1+\zeta)$. This root of unity has a valuation of 0, so $v_p\left(2\cos\frac{\theta}{2}\right) = v_p(1+\zeta) \geq \min\{v_p(1), v_p(\zeta)\} = 0$.

Now, consider the point $2C$. The x-coordinate has a valuation of

$$v_p\left(2 \cdot \frac{2\cos\frac{\theta}{2}}{x_1}\right) = v_p(2) + v_p(1+\zeta) - v_p(x_1) > 1 + 0 - 1 = 0$$

Analogously, we have the valuation of the y-coordinate is also positive, so $2C$ is necessarily red.

Consider opposite vertices on $P^\star$. Note that if A and B are opposite vertices, then $A + B = 2C$, and by Lemma 4.3, we have that A and B must be the same color.

Let $r = \frac{n}{2}$. Like before, we know that the points $(\frac{x_j}{x_1}, \frac{y_j}{y_k})$ are green, red, and blue for $j = n-1, 0, 1$ respectively. Thus, their opposite points, with $j = r-1, r, r+1$, must be colored green, red, and blue, respectively as well.

For all other points $(\frac{x_j}{x_1}, \frac{y_j}{y_k})$, we note $v_p(y_k) \geq v_p(y_j)$ so $v_p\left(\frac{y_j}{y_k}\right) \leq 0$. Therefore, these points cannot be red.

Consider $n_1(RB) - n_2(RB)$. This number is exactly 2, because it is 1 in the edge from $j = 0$ to $j = 1$, and 1 in the edge from $j = r$ to $j = r+1$. All other edges contribute 0 to this sum. Thus, by Sperner's Lemma in its integral form, we have that there must be a rainbow triangle in $P^\star$. □

Our theorem follows immediately. For all primes p, there must be a rainbow triangle in $P^\star$, which dictates that $v_p(m) \geq v_p(n)$. Thus, we have that n divides n. (It is always possible to dissect a regular n-gon into mn equal-area triangles. Inscribe the n-gon in a circle, and draw the segments from the center of the circle to each vertex, thus creating an equidissection with n triangles. Each triangle can easily be dissected into m triangles of equal area, so we've constructed mn total equal-area triangles.) Thus, we have proven the following:

Theorem 4.10 (Generalization of Monsky's Theorem)**.** *A regular n-gon with $n \geq 5$ has spectrum $\langle n \rangle$.*

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