# BORROMEAN RINGS, INTRANSITIVITY, AND OTHER ANALOGUES

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Abstract. When we consider examples of Brunnian links, we may think of the Italian House of Borromeo's coat of arms, Venn diagrams, and DNA representations. However, one interesting connection noted by mathematicians Brian Birgen [Bir15], Marc Chamberland, and Eugene A. Herman [CH15] was between Borromean rings and the popular game "Rock-Paper-Scissors," along with variants of its gameplay. Additionally, in their 2021 paper titled, "On Borromean links and related structures," O'Keeffe and Treacy [OT21] consider connections between Borromean rings, intransitive dice, and Rock-Paper-Scissors. This paper intends to pay homage to some of these interesting relationships.

## 1. A brief review of Borromean rings

First, let us recall the definition of knots and unknots.

**Definition 1.1.** A *knot* is an injective function from the circle $\mathbb{S}^1$ to $\mathbb{R}^3$. In other words, it is a map from a circle (the $n$-sphere in two dimensions) to the three-dimensional coordinate space. Two knots are considered to be equivalent if one can be continuously deformed into the other without any intersections. An *unknot* is a knot which can be continuously deformed to a circle.

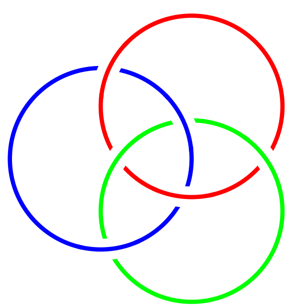

**Figure 1.** Borromean rings.

While it's easier to visualize knots in 3-dimensional space, we need to depict them in 2-dimensions in an comprehensible way. Thus, we can create a projection of the knot to a plane $\mathbb{R}^2$. We do so by creating a small gap to distinguish each crossing, as we can see in Figure 1. Note that the unknot essentially takes the form of a torus, which we see in Figure 2. One of the simplest non-trivial knots is called the *trefoil knot*, as shown in Figure 3. Some others non-trivial knots include the figure-eight knot and the cinquefoil knot.

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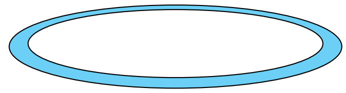

**Figure 2.** Unknot.

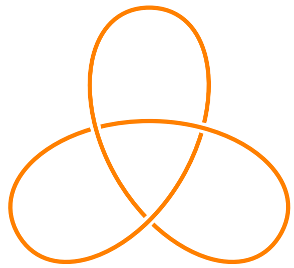

**Figure 3.** Trefoil knot.

Next, we would like to define a *link*:

**Definition 1.2.** A *link* is a finite collection of non-intersecting knots.

See the following diagram, showing a *Hopf link*.

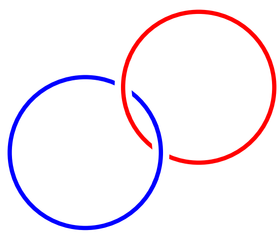

**Figure 4.** Hopf link.

Now we can give another important definition. First, note that "nontrivial" describes something that is linked (i.e., a link) and furthermore, a structure which is not an unknot.

**Definition 1.3.** A *Brunnian link* is a nontrivial link which, upon removal of any one of the link components, becomes an unlinked set of unknots.

This definition may be a bit hard to visualize, but the Borromean rings (considered to be Brunnian links) are a great example. If we remove any one of the rings, the rest of the structure falls apart into two unknots. Borromean rings are said to have a crossing number of 6, because there are 6 places where the rings cross over one another. Now that we've had a brief look at Borromean rings and Brunnian links, let's look a bit at the mathematics behind Rock-Paper-Scissors and the alternative version played with five game positions.

## 2. The Mathematics of Rock-Paper-Scissors and RPSLK

In this section, we will look at some of the mathematical properties of the popular game Rock-Paper-Scissors (RPS). First, let's take a look at how this game is played.

**Gameplay 2.1** (Rock-Paper-Scissors)**.** *The game is played with two players, and there are three symbols used in the game, namely Rock (denoted R), Paper (denoted P), and Scissors (denoted S). Rock beats Scissors ($R > S$), Scissors beats Paper ($S > P$), and Paper beats Rock ($P > R$). A single turn consists of both players simultaneously secretly selecting one of these and playing the symbol (via hand gesture) at the same time. If both players select the same symbol, repeat the process until one of the players' symbols "beats" the other in hierarchy, a winning move.*

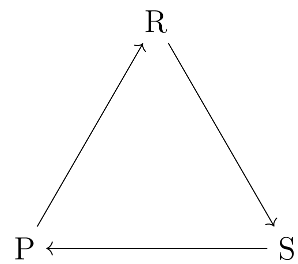


**Figure 5.** Three vertex directed graph representing Rock-Paper-Scissors.

Note that this game is considered fair, as neither player has an advantage over the other. This is because the symbol decision is secret and the playing of the symbol is simultaneous, so neither player is influenced by the other. Theoretically, the game is entirely luck-based, but in practice one can find psychological strategies based on their opponent's playing style. However, a more interesting version of RPS is Rock-Paper-Scissors-Lizard-Spock (RPSLK), which is a generalization of RPS to five symbols. We have the following gameplay.

**Gameplay 2.2** (Rock-Paper-Scissors-Lizard-Spock)**.** *The game RPSLK, created by Sam Kass and Karen Bryla, consists of the same three symbols as in RPS, as well as Lizard (L) and Spock (K). The hierarchy of the symbols is as follows.*

$$\begin{aligned} &R > S, && R > L.\\ &P > R, && P > K.\\ &S > P, && S > L.\\ &L > P, && L > K.\\ &K > R, && K > S. \end{aligned}$$

Note that this game is also fair, because the number of other moves a certain move wins against is the same as the number of moves it loses against. Figure 5 shows a representation of RPSLK as a five vertex directed graph.

One of the basic properties of RPSLK is that there are 24 different fair, five move games. Note that throughout this paper, we may use the terms position and move interchangeably.

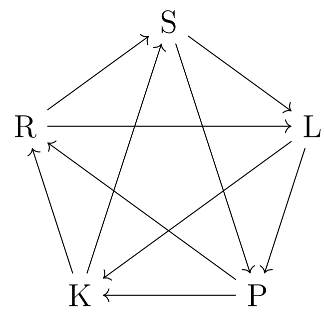


**Figure 6.** Five vertex directed graph representing RPSLK.

**Theorem 2.3.** *There are* 24 *different fair, five move games.*

*Proof.* Rather than letting each move correspond to the different positions in RPSLK, let's prove this statement for an arbitrary five move game with positions $A$, $B$, $C$, $D$, and $E$. This way, we can show that any five move game is isomorphic to RPSLK. We know that $A$ must beat two other positions, namely $B$ and $C$, which can be chosen in $\binom{4}{2} = 6$ ways. Then either $B > C$ or $C > B$; let's assume $B > C$. Because $A > C$ and $B > C$, then we must also have $C > D$ and $C > E$. Then either $D > E$ or $E > D$; let's assume $D > E$. Because $C > E$ and $D > E$, then we must also have $E > A$ and $E > B$. Since $A > B$ and $A > C$, we must have $D > A$ and $E > A$. Then $B > D$, so we have now found all the relations. There are $\binom{4}{2} \cdot 2 \cdot 2 = 24$ choices total, so this is the number of fair, five move games. This also follows for RPSLK (simply replace the variables in this proof with $R$, $P$, $S$, $L$, and $K$), and we have shown that all fair, five move games are isomorphic. ■

With more difficulty, it can also be shown that there are 570 fair, six move games. The proof follows a similar structure to the latter, but with more moves, the combinations are more complex.

## 3. Basics of Intransitive Dice

Now we will briefly shift our focus to intransitive dice (sometimes referred to as nontransitive dice). Typically, when given that $A > B$ and $B > C$, we have that $A > C$ by the transitive property. However, intransitive dice do not follow this property, hence their name.

**Definition 3.1.** A set of three dice, $A$, $B$, and $C$, is considered *intransitive* if the property holds that $A$ rolls higher than $B$ more than half the time, $B$ rolls higher than $C$ more than half the time, but $A$ does not roll higher than $C$ more than half the time. We denote this as $P(A > B) > \frac{1}{2}$.

*Example.* Consider the set of dice where the faces of each die are $A = [2, 2, 4, 4, 9, 9]$, $B = [1, 1, 6, 6, 8, 8]$, and $C = [3, 3, 5, 5, 7, 7]$. The probability that $A$ rolls higher than $B$, that $B$ rolls higher than $C$, and that $C$ rolls higher than $A$ are all $\frac{5}{9}$, so this set of dice is intransitive by definition. This is a particularly interesting example, because for each die, there is another that rolls higher more than half the time.

This is quite similar to the concept of Borromean rings being non-transitively-ordered. Like Borromean rings, if we remove one of the three dice, none of the required properties of intransitivity will be satisfied. Now that we understand all the basic concepts, let's look at some interesting connections between Borromean rings, Rock-Paper-Scissors/RPSLK, and intransitive dice.

## 4. Relationships between Borromean rings, RPSLK, and intransitive dice

Let's begin by looking at why three-component structures are so highly correlated to one another. We mustn't limit ourselves to seeing Borromean rings only as circular, as a 4-ring indicates a "ring" with four distinct edges in the form of a square. Note that the non-planar 4-rings share a common plane, but each ring is above one and under another. In the example, red (denoted $R$) is over green ($G$) and under blue ($B$). This is an intransitive relationship, and it turns out that we can imagine Borromean rings in yet another way, where each ring is outside one but inside another. From here, it's easy to compare the concept of Borromean rings with that of RPS (not necessarily RPSLK, which is not a three-component structure) and intransitive dice.

Here are some more interesting examples of three-component $n$-Borromean structures (see [OT21] for further information and diagrams).

*Example* (Braid structure). In this figure, we see that there is a clear $R > B > G > R$ relationship in the braid.

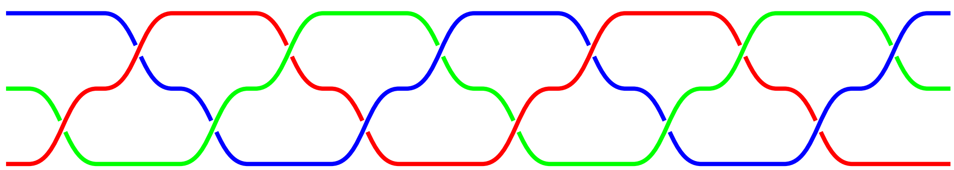

**Figure 7.** Braid structure.

*Example* (Kagome triaxial thread weave). In the Kagome triaxial thread weave, there are three sets of parallel threads in which no two sets are interwoven.

*Example* (3-Borromean chain mail). In the 3-Borromean chain mail, there are three different sets, where the rings in any two sets are not linked.

*Example* (Interwoven honeycomb). In the interwomen honeycomb, there is a pattern of three interwoven honeycomb nets in which no two pairs of nets are interwoven.

*Example* (3-periodic pattern). In the 3-periodic pattern, there is a one chain component of a 3-periodic pattern of three infinite sets of chains, where no two chains are linked.

Many detailed diagrams can be found for these examples in O'Keeffe and Treacy's paper.

## References

[Bir15] Brian J Birgen. The uniqueness of rock-paper-scissors-lizard-spock. *The College Mathematics Journal*, 46(4):270–273, 2015.

[CH15] Marc Chamberland and Eugene A Herman. Rock-paper-scissors meets borromean rings. *The Mathematical Intelligencer*, 37(06), 2015.

[OT21] Michael O'Keeffe and Michael MJ Treacy. On borromean links and related structures. *Acta Crystallographica Section A: Foundations and Advances*, 77(5):379–391, 2021.