THE ART GALLERY PROBLEM

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Abstract.
The Art Gallery Problem asks about the number of stationary guards necessary to guard a polygon shaped museum. We go over the Art Gallery Problem, proofs of the Art Gallery Theorem, and an extension of the problem to moving guards.

1. Defining the Problem

First, we formally define a museum and a guard.

Definition 1.1. Let a *museum* on n vertices be a simple polygon $\mathcal{A}$.

Definition 1.2. Define a *guard* O to be a point in some museum $\mathcal{A}$. If a segment OP for some point $P \in \mathcal{A}$ is fully contained within the museum, we consider the point P to be covered by O.

Definition 1.3. Let a museum $\mathcal{A}$ be considered *fully guarded* if every point P in it is covered by at least one guard O.

With these definitions, we can formally define the Art Gallery Problem.

The problem, originally formulated by Victor Klee in 1973, asked about the minimum number of guards necessary to guard a museum. This was answered by Vasek Chvátal in 1974, and the result later became known as Chv́atal's Art Gallery Theorem. Shortly after, in 1976, a simpler proof of the theorem was provided by Steve Fisk. Last, a book about the Art Gallery Problem was published in 1986 by Joseph O'Rourke, which explained various related problems and their solutions.

2. The Basic Problem

Theorem 2.1 (Chvátal's Art Gallery Theorem)**.** *Let $\mathcal{A}$ be a museum on n vertices. It is sufficient to have at most $\lfloor n/3 \rfloor$ guards in order for the museum to be fully guarded.*

We can first consider Fisk's proof of this result, which considers the triangulation of $\mathcal{A}$.[Fis78]

Proof.

Lemma 2.2. *We can triangulate any polygon into $n-2$ triangles using $n-3$ diagonals.*

Proof. We proceed by strong induction on n. Clearly this is true for a triangle, so we consider a polygon $\mathcal{P}$ with $n \geq 4$ vertices.

Let v_1 be some convex vertex of $\mathcal{P}$, and let v_2, v_3 be adjacent to it. We wish to split our polygon using some diagonal d into two smaller polygons. consider $d = v_2v_3$. If this segment is contained entirely within the polygon, then we can consider the two polygons $\mathcal{P}_1$ and $\mathcal{P}_2$

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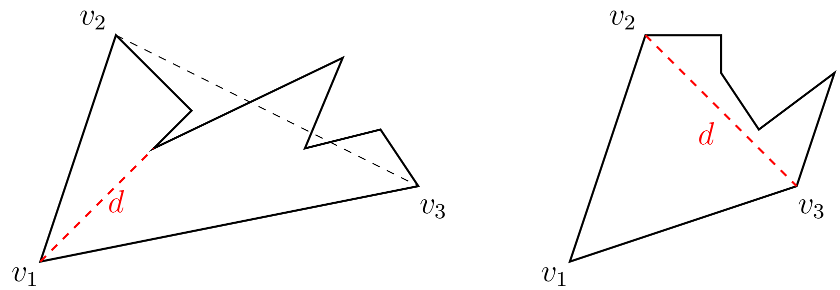

Figure 1. Possible constructions of the diagonal d.

which this diagonal splits $\mathcal{P}$ into. Otherwise, consider a point v_4 which is closest to v_1, and use $d = v_1v_4$ to split the polygon (see Figure 1).

Letting $\mathcal{P}_i$ have n_i vertices, we find that $n_1 + n_2 = n + 2$, as the diagonal d is included in both $\mathcal{P}_i$. We find that $n_i \geq 3$ and thus $n_i < n$, so using the inductive hypothesis on $\mathcal{P}_1, \mathcal{P}_2$ gives that $\mathcal{P}$ can be triangulated into $(n_1 - 2) + (n_2 - 2) = n - 2$ triangles using $(n_1 - 3) + (n_2 - 3) + 1 = n - 3$ diagonals. ■

We next consider some properties of this triangulation graph.

Definition 2.3. Define the *dual graph* of $\mathcal{A}$ to be the graph constructed using one node for each triangle and edges connecting triangles which share a diagonal.

The dual graph has some properties which we can use to find some properties of our original triangulation graph.

Lemma 2.4. *The dual graph of $\mathcal{A}$, is a tree for which each node has a degree no more than* 3.

Proof. First, assume the graph were to have a cycle. Then there would be some exterior point enclosed by $\mathcal{A}$, which contradicts the definition of a polygon. Thus the graph must be a tree, and the degree condition follows from the fact that a triangle has 3 sides. ■

Finally, we use properties of the dual graph to deduce properties of our original triangulation graph. We first consider ears of our museum $\mathcal{A}$.

Definition 2.5. Define an ear of $\mathcal{A}$ to be a set of three consecutive vertices v_1, v_2, v_3 in this order, such that v_1v_3 is fully contained within $\mathcal{A}$.

We can use Lemma 2.4 to prove a powerful result relating to ears of the graph, known as Meister's Two Ears Theorem.

Theorem 2.6 (Meister's Two Ears Theorem)**.** *The museum $\mathcal{A}$ on $n \geq 4$ vertices has at least two non-overlapping ears.*

Proof. Each ear corresponds to a leaf of the dual graph, and since the graph is a tree for which each node has degree no more than 3, it must have at least two leaves. ■

With all of these properties, we can finally prove a powerful result relating to the triangulation graph.

Lemma 2.7. *The graph produced upon triangulation of the museum $\mathcal{A}$ is* 3*-colorable.*

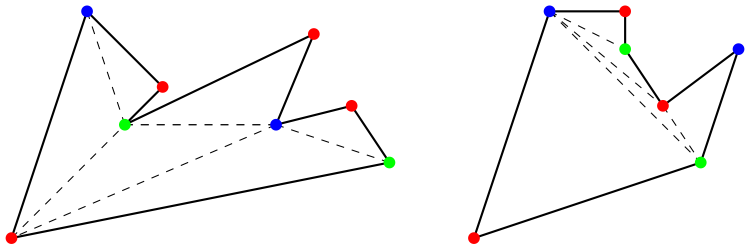

Figure 2. 3-colorings of triangulated museums.

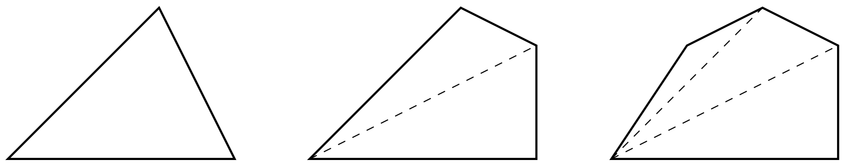

Figure 3. Museums on 3, 4, and 5 vertices are all fans.

Proof. We proceed by induction on n. It is trivial to see that a triangle is 3-colorable. Now assume that an arbitrary museum on n vertices is 3-colorable. We can add an ear to this graph, connecting it to a diagonal v_1v_2 for vertices v_1, v_2. These vertices must be colored differently, say with colors a and b. Then the third vertex of the ear v_3 can be colored with color c. This produces a valid coloring on $n+1$ vertices. Thus the graph produced upon triangulation of $\mathcal{A}$ must be 3-colorable (See Figure 2 for examples). ■

With this fact, we complete Fisk's proof; for any museum $\mathcal{A}$, we triangulate it, 3-color it, and consider the color a that the least number of vertices are colored with. Letting T_a be the number of vertices colored with color a, we find that $T_a \leq \lfloor n/3 \rfloor$ (as T_a is an integer). Since all triangles are convex, the points colored a cover the entire museum. Thus it is always sufficient to have $\lfloor n/3 \rfloor$ guards. ■

Next, we consider a similar proof by Chvátal [Chv75]. This proof uses the same triangulation idea, but uses induction to prove the result.

Proof.

Definition 2.8. Define a *fan* to be a triangulation of some subgraph S of $\mathcal{A}$ such that all triangles share some vertex v, which we call the *fan center*.

Proposition 2.9. *Every triangulation of a museum $\mathcal{A}$ on n vertices can be partitioned into $x \leq \lfloor n/3 \rfloor$ fans.*

Our base cases are $n \in \{3, 4, 5\}$, which all result in fans, as shown in Figure 3. The inductive step follows from a bit of casework that we will not describe, but we find that we can always partition our polygon into $x \leq \lfloor \frac{n}{3} \rfloor$ fans. We place a guard on the fan center of each of these fans, so we need at most $\lfloor \frac{n}{3} \rfloor$ guards to guard the museum $\mathcal{A}$. ■

3. Moving Guards

Consider an altered version of the same problem, except guards may move along line segments. If a point is covered by some point on any of the line segments, we consider this point to be covered by a guard.

Theorem 3.1 (O'Rourke [ORo97])**.** *It is sufficient to have $\lfloor n/4 \rfloor$ moving guards to guard a museum $\mathcal{A}$.*

Definition 3.2. Define a *diagonal guard* to be a guard placed on some edge of the triangulation of $\mathcal{A}$.

Definition 3.3. Let E be an edge of $\mathcal{A}$. The *contraction* of E is the replacement of the endpoints of E with a node x such that x is adjacent to all nodes that the endpoints were adjacent to.

We can use this operation to derive an interesting result relating to the triangulation of $\mathcal{A}$.

Lemma 3.4. *The graph resulting from a contraction of the edge E of the triangulation graph G of $\mathcal{A}$ is a triangulation graph of some museum $\mathcal{B}$.*

Proof. Let Q be the planar graph corresponding to G, and let E connect vertices x and y. Let the adjacent vertices to x be $x_1, ..., x_i$ and let the adjacent vertices to y be $y_1, ..., y_j$. Consider a vertex v on E. First, we connect x_1 to v, and remove the diagonal (x, x_1). Then connect x_2 to v using some curved edge such that the edge remains in the region bounded by x, v, x_1, and x_2, and then remove the diagonal (x_2, x). Continue this for the rest of the x_i, then perform a similar process with the y_i. Next, apply Fary's theorem, which states that every planar graph with possibly curved edges maps to some planar graph with straight edges. This transformation results in a museum $\mathcal{B}$. ■

We use this lemma to produce a more powerful statement about the position of guards in the museum $\mathcal{A}$.

Lemma 3.5. *Assume $f(n)$ diagonal guards are necessary to cover an n node triangulation graph. Then $f(n-1)$ diagonal guards are necessary to cover an n-node triangulation graph G with a guard at one of its vertices.*

Proof. Consider the vertex x which the guard is at. Contract some edge E of the graph connecting x with another node y such that E is an edge of the museum $\mathcal{A}$. Due to Lemma 3.4, we find that this produces some triangulation graph G' of a museum $\mathcal{B}$ on $n-1$ vertices. Then this graph can be covered by $f(n-1)$ diagonal guards.

Consider the vertex z that replaces x and y in the graph G', and assume that no guard is placed at z. Then the same guards will also cover the museum $\mathcal{A}$, as there is a guard at the vertex x and all triangles not covered by the guard at x are covered by some guards which cover the graph G'. If a guard is placed at z when covering G', we can move it to x, causing the museum $\mathcal{A}$ to remain fully guarded.

■

Lemma 3.6. *If $\mathcal{A}$ has $n \geq 10$ vertices, there is a diagonal in the triangulation graph of $\mathcal{A}$ which partitions the graph into two pieces each containing $5, 6, 7,$ or 8 arcs corresponding to edges of $\mathcal{A}$.*

Proof. Let D be a diagonal guard which partitions off $k \geq 5$ edges. Label the vertices 0 through n such that D is the diagonal $(0, k)$. Then the diagonal supports some triangle such that the vertex opposite d is at x such that $0 \leq x \leq k$. Then $x \leq 4$ and $k - x \leq 4$, so $5 \leq k \leq 8$. Examples of possible diagonals are shown in Figure 4.

■

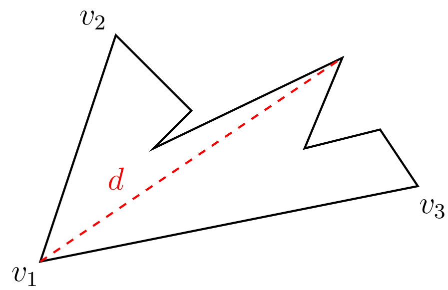

Figure 4. Diagonals d which partition off at least 4 vertices.

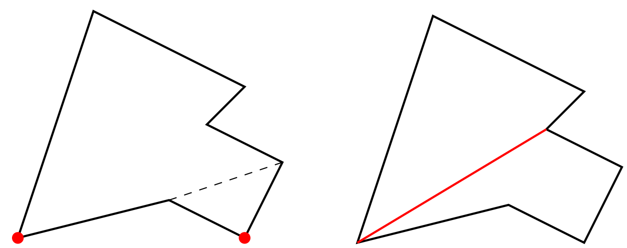

Figure 5. Vertex and diagonal guards guarding a simplified version of the Louvre.

Proof of Theorem 3.1.

Notice first that the theorem is true for $5 \leq n \leq 9$, which will serve as our base case.

Now assume that the theorem holds for $z < n$. We know from Lemma 3.6 that there exists some diagonal D that partitions the triangulation graph G of $\mathcal{A}$ into two graphs G_1 and G_2 such that G_1 has $4 \leq k \leq 8$ edges of G. We consider cases for k.

Case 1: ($k \in \{5, 6\}$) Since G_1 has k boundary edges including D, it can be guarded by a singular diagonal guard. Now consider G_2. It has $n - k + 1 \leq n - 5 + 4 = n - 4$ edges including D, so it can be covered by $\lfloor n/4 \rfloor - 1$ guards. Thus we find that we need a total of $\lfloor n/4 \rfloor$ diagonal guards to fully guard the museum $\mathcal{A}$. ■

Case 2: ($k = 7$) We find that either $(0, 3, 7)$ or $(0, 4, 7)$ is bounded by d in G_1. By symmetry, these cases are equivalent, so we can assume that $(0, 3, 7)$ is bounded by d in G_1. We find that the quadrilateral $(0, 1, 2, 3)$ has 2 distinct triangulations, both of which result in sufficiency of $\lfloor n/4 \rfloor$ guards. ■

Case 3: ($k = 8$) Notice that G_1 has 9 boundary edges, so it can be covered by 2 guards, with one at vertex 0. Then G_2 has $n - 7$ edges, but since there is a guard at vertex 0, we only need $f(n - 8)$ guards to cover it, due to Lemma 3.5. Then we need $\lfloor (n - 8)/4 \rfloor = \lfloor n/4 \rfloor - 2$ guards to cover G_2, which combined with the 2 guards necessary to guard G_1 yields $\lfloor n/4 \rfloor$ guards. ■

With these cases, we complete the proof of sufficiency of $\lfloor n/4 \rfloor$ diagonal guards to fully guard the museum $\mathcal{A}$. ■

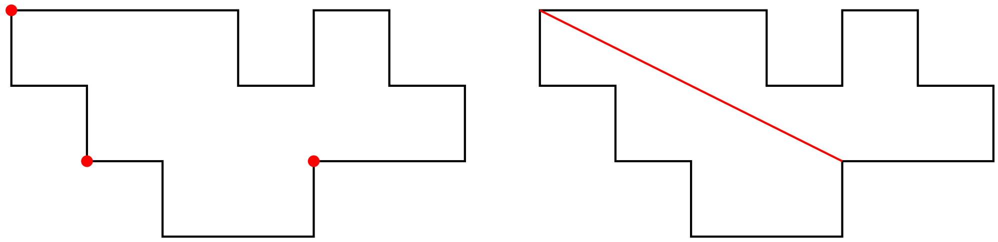

Figure 6. Vertex and diagonal guards guarding a simplified version of the White House.

4. Museums and Extensions

Consider applying the Art Gallery Theorem to some real museums. As shown in Figure 5, $\lfloor 7/3 \rfloor = 2$ vertex guards and $\lfloor 7/4 \rfloor = 1$ diagonal guards are sufficient to fully guard a simplified version of the Louvre in Paris.

Similarly, as shown in Figure 6, we only need 3 vertex guards and 1 diagonal guard to guard a simplified version of the White House, since our upper bound is not always necessary.

We find that the White House is a special type of museum, called an *orthogonal museum*, since it has only 90 degree angles. Orthogonal museums require only $\lfloor n/4 \rfloor$ vertex guards to be fully guarded and $\lfloor (3n+4)/16 \rfloor$ diagonal guards to be fully guarded, although we will not provide a proof of this fact. A proof is provided by O'Rourke in [ORo97].

5. Conclusion

We went over the Art Gallery Theorem, and two of its proofs, provided by Fisk and Chvátal. We went on to extend our problem to moving guards, and finished by looking at some diagrams and examples of possible placements of guards in famous museums.

References

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