Number of Spanning Trees in a Graph

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1 Introduction

Spanning trees are an important topic of study in graph theory. On the surface level, they are useful in determining how to connect different cities together, but they can also be used in connecting computer networks and electrical grids. Today we will explore how to count the number of spanning trees in a graph or system. Let us begin with some definitions:

Definition 1 *A graph* $G = (\mathcal{V}, \mathcal{E})$*, where* $\mathcal{V}$ *is the set of vertices in the graph and* $\mathcal{E}$ *is the set of edges in the graph.*

Throughout this paper, we will let n be the number of nodes in a given graph and m be the number of edges. One important structure we can define given a graph G is a spanning tree, $\mathcal{S}$.

Definition 2 *A spanning tree* $\mathcal{S}$ *is a set of* $n-1$ *edges such that for any two nodes,* $i, j \in \mathcal{V}$*, there is some sequence of nodes* $v_1, v_2, \ldots, v_k \in V$ *such that for all* t *where* $1 \leq t \leq k-1$*, there is an edge from* v_t *to* v_{t+1} *in* $\mathcal{S}$*. In other words, a spanning tree is a minimal collection of edges that connects all nodes in* $\mathcal{V}$*.*

In this paper, we will explore Kirchhoff's Theorem, which tells us the number of spanning trees in a graph.

2 Kirchhoff's Theorem

Before we can introduce Kirchhoff's Theorem, there is some more notation to introduce:

Definition 3 *The degree matrix* D *of a graph* G *is an* n *by* n *matrix, where* $D_{ij} = 0$ *if* $i \neq j$ *and* $D_{ii} = |i|$*, where* $|i|$ *represents the degree of node* i*.*

Definition 4 *The adjacency matrix* A *of a graph* G *is an* n *by* n *matrix where* $A_{ij} = 1$ *iff there exists an edge connecting* i *and* j*, and* $A_{ij} = 0$ *otherwise.*

Definition 5 $M[s]$ *denotes the matrix* M *with its* s*th row and column removed.*

Definition 6 *Let $c(G)$ denote the number of spanning trees in G.*

Definition 7 *$G\backslash i$ denotes the graph G with the node i removed, and $G\backslash e$ denotes the graph G with the edge e removed. Furthermore, G/e represents the graph G with the edge e contracted. In other words, the two vertices the edges combined are merged into one vertex, with all the edges from both original vertices.*

Finally, for convenience, we will let $L_G = D - A$. This is known as the Laplacian of G. Let us now introduce Kirchhoff's Theorem:

Theorem 1 *For all s where $1 \leq s \leq n$, $det\left((D - A)[s]\right) = c(G)$.*

In English, the determinant of the Laplacian with any specific row and column removed gives the number of spanning trees in G.

3 Proof of Kirchhoff's Theorem

We will prove this theorem using induction on our edges.

3.1 Base Case

Our base case will be any graph with isolated vertices.

We will require the follow lemma:

Lemma 1 $det(D - A) = 0$

We know that if any row in a matrix can be written as a linear combination of other rows in that matrix, the matrix has a determinant of 0. Note that in $D - A$, every row and column has a sum of 0. If M_k indicates the kth row of matrix M, let R be row vector where:

$$R = \sum_{k=1}^{n-1} (D - A)_k$$

Then, since the column sums are 0, we have that $(D - A)_n = -R$. Thus, we have written a row as a linear combination of other rows, and shown Lemma 1.

Furthermore, we will take the following as given:

Lemma 2 *If E_{ii} is an n by n matrix with a 1 at i, i and 0's everywhere else:*

$$det\left(A + E_{ii}\right) = det(A) + det\left(A[i]\right)$$

Some simple intuition for this lemma lies in the row-expansion definition of the determinant. Take the matrix of cofactors C, where C_{ij} is the determinant of matrix C with the ith row and jth column removed, then multiplied by $(-1)^{i+j}$. Then, for any row i,

$$det(M) = \sum_{j=1}^{n} M_{ij}C_{ij}$$

If we look at how this changes when we increment a specific element, we can see that it simply adds the cofactor the determinant, which is what we see in our original equation.

Let i be our isolated node. Then, let $G' = G\backslash i$. Clearly, $(D' - A') = (D - A)[i]$. Thus,

$$det\left((D - A)[i]\right) = det\left(D' - A'\right) = 0$$

by Lemma 1. Since i is isolated, 0 is indeed the number of spanning trees in G.

3.2 Inductive Step

We will prove this theorem for a general i where i denotes the row and column we are removing from G. Due to our base case, we can assume that there exists some base case $e = (i, j)$.

The following idea will be the basis for our induction:

$$c(G) = c(G/e) + c(G\backslash e)$$

In one case, we are forced to use e by contracting it, and in the other, we exclude e entirely.

Now, consider the difference between $L_G[i]$ and $L_{G\backslash e}[i]$. The only difference, is that $(L_G[i])_{jj} = 1 + \left(L_{G\backslash e}[i]\right)_{jj}$. In other words, $L_G[i] = E_{jj} + L_{G\backslash e}[i]$. Then, applying Lemma 2:

$$\begin{aligned} det\left(L_G[i]\right) &= det\left(L_{G\backslash e}[i] + E_{jj}\right) \\ &= det\left(L_{G\backslash e}[i]\right) + det\left(L_{G\backslash e}[i][j]\right) \end{aligned}$$

However, when we remove both the ith and jth rows and columns, the edge e between them is insignificant. Thus, $L_{G\backslash e}[i][j] = L_G[i][j]$. Furthermore, consider $L_{G/e}[i]$. Since we have contracted e, i and j now represent the same node. Thus, $L_{G/e}[i] = L_G[i][j]$. Then:

$$\begin{aligned} det\left(L_{G\backslash e}[i]\right) + det\left(L_{G\backslash e}[i][j]\right) &= det\left(L_{G\backslash e}[i]\right) + det\left(L_G[i][j]\right) \\ &= det\left(L_{G\backslash e}[i]\right) + det\left(L_{G/e}[i]\right) \\ &= c(G\backslash e) + c(G/e) \\ &= c(G) \end{aligned}$$

concluding the proof.