Determinants and Path Counting

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1 Lindström–Gessel–Viennot Lemma

Let G be a directed acyclic graph with n designated origin and destination nodes, and let A be the $n \times n$ matrix whose (i, j)-entry is the number of paths from the ith origin vertex to the jth destination vertex. The following statements hold:

1. The number of n-paths is equal to the perm(A).
2. The determinant of A equals $Even(G) - Odd(G)$ where $Even(G)$ is the number of non-intersecting n-paths corresponding to even permutations and $Odd(G)$ is the number of non-intersecting n-paths corresponding to odd permutations.

Recall that directed means that the edges on the graph G have a direction that must be followed. Acyclic means that we cannot return to our starting point walking on the edges and following the directions assigned to those edges. Non-intersecting paths do not share vertices.

The lemma can be summarized as

$$det(A) = \sum_{P} sgn(\sigma(P)) \prod_{i=1}^{n} w(P_i)$$

where $w(P_i)$ is the wight of the path.

1.1 Example

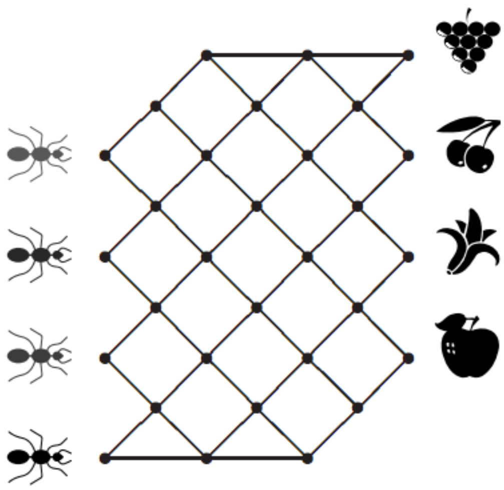

Figure 1: Ants and food. Week 7, problem 5

Question 1: Simultaneously, how many ways can can the ants reach the different types of food?[1]

Let an n-path to be a collection of paths from a set of n origin vertices to a set of n destination vertices. We can use a method similar to pascal's triangle to figure out in how many ways each ant can

[1] https://www.math.hmc.edu/ benjamin/papers/monthly481-492.pdf

get to each type of food. We can write this information in a matrix whose (i, j)-entry is the number of ways that ant i can reach food j.

$$A = \begin{bmatrix} 14 & 6 & 1 & 0 \\ 20 & 15 & 6 & 1 \\ 15 & 20 & 15 & 6 \\ 6 & 15 & 20 & 14 \end{bmatrix}$$

We can begin answering our question by figuring out the number of ways ant i can get to food i. This is $a_{11}a_{22}a_{33}a_{44}$. Now we want to consider all the possible ways for ant i to get to food j. First, we can look at the permutation σ of $\{1, 2, 3, 4\}$. For each permutation σ, there are $a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)}a_{4,\sigma(4)}$ paths. We want to sum these paths so we can take

$$\sum_{\sigma \in S_4} a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)}a_{4,\sigma(4)}$$

That is also known as perm(A). This gets us the answer 171361.

Question 2: In how many ways can the ants get from the origin vertices to the food, so that each ant gets a different type of food, and no two ant paths cross?

We know by the lemma we can just compute the determinant of A. Determinant of A:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)}a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

S_n is the permutations of $1, 2, 3, \cdots, n$. $\operatorname{sgn}(\sigma) = 1$ when σ is an even permutation while $\operatorname{sgn}(\sigma) = -1$ when σ is an odd permutation. We get 889 as the answer.

1.2 Catalan Numbers

Recall week 7, problem 6, which talked about Catalan numbers. We said that C_n is the number of paths from $(0, 0)$ to (n, n) that never cross the line $y = x$ that take unit steps towards north or east.

Let A be the $(n+1) \times (n+1)$ matrix. If we look at the figure, the number of ways to get from vertex O_i to D_j is C_{i+j}. Therefore, let each (i, j) of matrix A entry be C_{i+j}. There is only one way to create $n + 1$ non-intersecting paths, so the $\det(A)$ is 1.

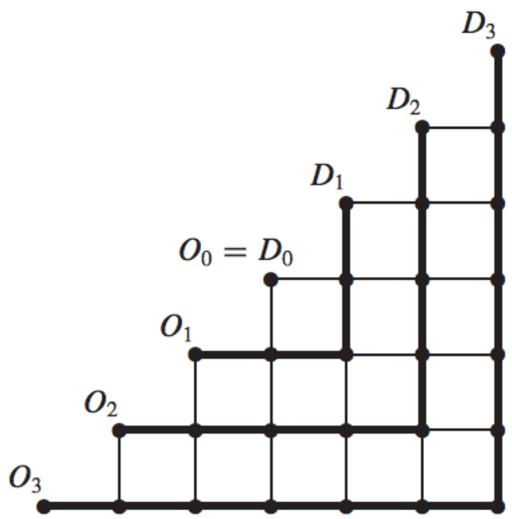

Figure 2: Week 7, problem 6

2 Schur Polynomials

2.1 Definition One of Schur Polynomials

Definition 2.1. For a partition λ, a semi-standard young tableaux (SSYT) of shape λ is a filling of the basis of *lambda* with positive integers so that the rows weekly increase and the columns strictly increase.

Example 2.1.1. For $\lambda = (4, 2, 1)$, a SSYT of shape λ is

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

Definition 2.2. If T is a SSYT, then we write

$$x^T = x_1^{number of 1's} + x_2^{number of 2's} \cdots$$

Example 2.2.1. We have

$$x^{\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}} = x_1^2 x_2^2 x_3 x_5 x_6$$

Definition 2.3. We define the Schur polynomial of shape λ to be

$$s_\lambda(x_1, x_2, \cdots) = \sum_{T \text{ of shape } \lambda} x^T$$

Example 2.3.1. Consider s_{21}. We could have

$$\begin{array}{|c|c|}\hline i & i \\ \hline j \\ \cline{1-1}\end{array} \text{ or } \begin{array}{|c|c|}\hline i & j \\ \hline j \\ \cline{1-1}\end{array} \quad \text{for } i < j$$

or

$$\begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array} \text{ or } \begin{array}{|c|c|}\hline i & k \\ \hline j \\ \cline{1-1}\end{array} \quad \text{for } i < j < k$$

Therefore, we have

$$s_{21} = \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2 + \sum_{i<j<k} x_i x_j x_k$$

2.2 Definition Two of Schur Polynomials

Definition 2.4. P is a symmetric polynomial if for any permutation σ of the subscripts of the variables on has $P(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}) = P(x_1, x_2, \cdots, x_n)$

Example 2.4.1. The following polynomials are symmetric:

- $X_1 + X_2$
- $4X_1X_2X_3 - 3X_1X_2 - 3X_1X_3 - 3X_2X_3$

The following polynomials are not symmetric:

- $X_1 - X_2$
- $2X_1X_2X_3 - 3X_1X_2 - 4X_1X_3 - 5X_2X_3$

Definition 2.5. The complete homogeneous symmetric polynomial of degree k and in n variables

$$X_1, \cdots, X_n$$

written h_k is the sum of all monomials of total degree k in the variables:

$$h_k(X_1, X_2, \cdots, X_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} X_{i_1} X_{i_2} \cdots X_{i_3}$$

Example 2.5.1. For $n = 1$

- $h_1(X_1) = X_1$

For $n = 2$

- $h_1(X_1, X_2) = X_1 + X_2$
- $h_2(X_1, X_2) = X_1^2 + X_1X_2 + X_2^2$

Definition 2.6. $s_\lambda(x_1, x_2, \cdots, x_n) = \det((h_{\lambda_i+j-i})_{i,j}^{r\times r})$

Example 2.6.1.

$$s_{(3,2,2,1)} = \begin{vmatrix} h_3 & h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 \end{vmatrix}$$

2.3 Proof

Given any partition λ, consider r origin nodes at $a_i = (r + 1 - i, 1)$ and the r destination nodes $b_i = (\lambda_i + r + 1 - i, n)$ as points in the lattice $\mathbb{Z}^2$ which can be a directed graph if the only allowed the directions are north and east. The weight associated with each horizontal edge at height i is x_i and the weight of each vertical edge is 1. With this definition, r-tuples of the paths from A to B are SSYT of shape λ and the weight of such an r-tuple is the corresponding summation in the first definition of Schur polynomials. On the other hand, the matrix M is the same as what we get from this, so we get the desired equality. [2]

[2]https://en.wikipedia.org/wiki/LindstromGesselViennotlemma