When are binomial coefficients perfect powers?

Rasika Iyer

1 Introduction

When is $\binom{n}{k}$ equal to m^l? It is fairly simple to see that there are infinitely many solutions when $k = l = 2$. That is, $\binom{n}{2} = m^2$. From here, we know

$$\begin{aligned} n(n-1) &= 2m^2 \\ (2n-1)^2((2n-1)^2-1) &= 2m^2 \\ (2n-1)^2 4n(n-1) &= 2m^2 \\ (2(2m(2n-1))^2 &= 2m^2 \\ \binom{(2n-1)^2}{k} &= (2m(2n-1))^2. \end{aligned}$$

Starting with $\binom{9}{2} = 6^2$, we obtain infinitely many solutions, with the next one being $\binom{289}{2} = 204^2$. However, this does not yield all the solutions. For example, $\binom{50}{2} = 35^2$ starts another sequence, as does $\binom{1682}{2} = 1189^2$. For $k = 3$, it is known that $\binom{n}{3} = m^2$ has the unique solution $n = 50$ and $m = 140$. But now, we are at the end of the line. For $k \geq 4$ and $l \geq 2$, no solutions exist. We will prove that this is true for $l = 2$. An extension is required to prove that $l \geq 3$, but the crux of the proof lies below.

2 Binomial coefficients are (almost) never powers

Theorem 2.1. *For $n \geq 2k$, the equation $\binom{n}{k} = m^l$ has no integer solution with $l \geq 2$ and $4 \leq k \leq n-4$.*

Proof. Suppose the theorem is false, and that $\binom{n}{k} = m^l$. We begin with, a strengthening of Bertrand's postulate:

Lemma 2.1 (Sylvester's Theorem)**.** *If $n \geq 2k$, then at least one of the numbers n, $n-1$,..., $n-k+1$ has a prime divisor p greater than k.*

Note that for $n = 2k$ we obtain precisely Bertrand's postulate.

Remark 2.1. There is an equivalent of stating Sylvester's Theorem: For $n \geq 2k$, the binomial coefficient

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

always has a prime factor $p > k$.

By Lemma 2.1, there is a prime factor p of $\binom{n}{k}$ greater than k, hence p^l divides $n(n-1)\dots(n-k+1)$. Clearly, only one of the factors $n-i$ can be a multiple of p (because of $p > k$), and we conclude $p^l | n-i$, and therefore

$$n \geq p^l > k^l \geq k^2 \tag{1}$$

Consider any factor of the numerator written in the form $n-j = a_j m_j^l$, where a_j is not divisible by any nontrivial l-th power.

Lemma 2.2. *If $i = j$, then $a_i = a_j$.*

Proof. Assume $a_i = a_j$ for some $i < j$. Then, $m_i \geq m_j + 1$ and

$$\begin{aligned} k &> (n-i) - (n-j) = a_j(m_i^l - m_j^l) \geq a_j((m_j+1)^l - m_j^l) \\ &> a_j l m_j^{l-1} \geq l(a_j m_j^l)^{1/2} \geq l(n-k+1)^{1/2} \\ &\geq l(\frac{n}{2}+1)^{1/2} > n^{1/2} \end{aligned}$$

which contradicts $n > k^2$ from Equation 1. ∎

Next, we need to prove that the a_i's are the integers $1, 2, \ldots, k$ in some order. Since, by Lemma 2.2, they are all distinct, it suffices to prove that

$$a_0 a_1 \ldots a_{k-1} \text{ divides } k!.$$

Substituting $n - j = a_j m_j^l$ into the equation $\binom{n}{k}$, we obtain

$$a_0 a_1 \ldots a_{k-1}(m_0 m_1 \ldots m_{k-1})^l = k! m^l$$

Cancelling out the common factors between $m_0 m_1 \ldots m_{k-1}$ and m yields

$$a_0 a_1 \ldots a_{k-1} u^l = k! v^l$$

with $gcd(u, v) = 1$. It remains to show $v = 1$.If not, then v contains a prime divisor p. Since $gcd(u, v) = 1$, p must be a prime divisor of $a_0 a_1 \ldots a_{k-1}$ and hence is less than or equal to k.

Lemma 2.3 (Legendre's theorem)**.** *The number $n!$ contains the prime factor p exactly*

$$\sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor$$

times.

By Lemma 2.3, we know that $k!$ contains p to the power $\sum_{i \geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor$. We now estimate the exponent of p in $n(n-1)\ldots(n-k+1)$. Let i be a positive integer, and let $b_1 < b_2 < \cdots < b_s$ be the multiples of p^i among $n, n-1, ..., n-k+1$. Then $b_s = b_1 + (s-1)p^i$ and hence

$$(s-1)p^i = b_s - b_1 \leq n - (n-k+1) = k-1,$$

which implies

$$s \leq \left\lfloor \frac{k-1}{p^i} \right\rfloor + 1 \leq \left\lfloor \frac{k}{p^i} \right\rfloor + 1.$$

So for each i the number of multiples of p^i among $n, \ldots, n-k+1$, and hence among the a_j's, is bounded by $\left\lfloor \frac{k}{p^i} \right\rfloor + 1$. This implies that the exponent of p in $a_0 a_1 \ldots a_{k-1}$ is at most

$$\sum_{i \geq 1}^{l-1} \left(\left\lfloor \frac{k}{p^i} + 1 \right\rfloor \right).$$

Taking both counts together, we find that the exponent of p in v^l is at most

$$\sum_{i \geq 1}^{l-1} \left(\left\lfloor \frac{k}{p^i} + 1 \right\rfloor \right) - \sum_{i \geq 1}^{l-1} \left\lfloor \frac{k}{p^i} \right\rfloor \leq l - 1,$$

and we have our desired contradiction, since v^l is an l-th power. Indeed, since $k \geq 4$ one of the a_i's must be equal to 4, but the a_i's contains no squares.

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