OF FRIENDS AND POLITICIANS

RAJIV NELAKANTI

1. Introduction

Here's the premise: we have a group of people, such that every two people share a mutual friend. Upon inspection, we see that there always appears to be a politician: a person that is friends with everybody. To mathematically state this, we turn to graph theory.

2. The Theorem

Theorem 2.1. *Suppose that G is a finite graph in which any two vertices have precisely one common neighbor. Then there is a vertex which is adjacent to all other vertices.*

Note: It is helpful to note that any graph with the desired properties is a windmill graph; that is, any such G can be thought of as a bunch of triangles, all adjoined at a common vertex to create a windmill. As you follow along the proof, keep the image of a windmill in mind to understand the properties of this class of graphs.

Proof. The proof I will provide is due to Paul Erdos, Alfred Rényi, and Vera Sós. Proof by contradiction. Suppose there exists some graph G with no vertex adjacent to all others. From here, the proof falls into two parts: one combinatorics, and the other linear algebra.

Part 1: Combinatorics. We begin by asserting that the C_4-condition is satisfied, which means that no 4-cycle exists in G. This is obvious, because a 4-cycle implies there exists a pair of vertices with more than one mutual neighbor (draw out and inspect a 4-cycle to visualize this).

Lemma 2.2. *Given the assumption, G is a regular graph; that is, $d(u) = d(v)$ for any $u, v \in V$*

Proof. First, we will prove this for nonadjacent u, v. Let $d(u) = k$, so that u has neighbors $w_1, w_2, \ldots, w_k$. By the common neighbor property, there exists some w_i adjacent to v, say w_2, and some w_j adjacent to w_2, say w_1. Vertex v has w_2 as common neighbor to w_1, and for $i \geq 2$, has z_i as common neighbor to w_i. Each z_i is distinct by the C_4-condition, which implies that $d(v) \geq d(u)$. Symmetrically, $d(u) \geq d(v)$, and thus $d(u) = d(v)$.

Thus, every vertex not adjacent to either u or v also has degree k. The only vertex remaining to prove has degree k is w_2. But, as assumed, since there is no politician in G, w_2 has a non-neighbor, forcing its degree to be k as well. ■

Using the principle of inclusion-exclusion, we can conclude that the number of vertices in G is $k^2 - k + 1$. We can note that $k \geq 2$ because this equation forces G to be K_1 and K_3 for $k = 1$ and 2 respectively, both of which are graphs with a politician.

Date: April 4, 2017.

Part 2: Spectral Theory. Let A be the adjacency matrix of G; that is, the value of the i^{th} row and the j^{th} column is 1 if vertices g_i and g_j are adjacent and 0 otherwise. By the properties of G, we see that A^2 is the matrix of all 1's, except for the diagonal from the upper left to the lower bottom, which is filled with only k. We get that $A^2 = (k-1)I + J$, where I is the identity matrix and J is the matrix of only 1's. Through spectral voodoo, we get that A^2 has eigenvalues k^2, with multiplicity 1, and $k-1$, with multiplicity $n-1$. A is pretty enough (symmetric and diagonizable) so that we can conclude that its eigenvalues are k and $\pm\sqrt{k-1}$. Of the $n-1$ eigenvalues of the latter, suppose r are $\sqrt{k-1}$ and s are $-\sqrt{k-1}$. Again, spectral magic gives us $k + r\sqrt{k-1} - s\sqrt{k-1} = 0$, since the sum of the eigenvalues of A is equal to its trace. We get $\sqrt{k-1} = \frac{k}{s-r}$. Since the square root of a natural number is equal to a rational number, it is an integer, say h (this should be pretty obvious). Then, $h(s-r) = h^2+1$. Since h divides h^2+1, h must be 1, so $k=2$, which is a contradiction, as shown earlier. Whew! ■

3. Generalizations

We can look at this problem in a more general sense by rephrasing the theorem as such: Suppose G is a graph with the property that there is exactly one path of length $l=2$ between any two vertices. Clearly, this is an equivalent statement, but it is in a more general form, because it begs the question of what such a graph with larger l might look like.

In fact, we have a whole conjecture that proposes that no graph with larger l can exist, attributable to Anton Kotzig.

Conjecture 3.1. *Kotzig's Conjecture There are no finite graphs such that between any two vertices, there is exactly one path of length l, for all $l > 2$.*

Kotzig and others have checked this conjecture up to $l = 33$ (painstakingly, I can only imagine), but a general proof evades our grasp yet.