3 PROOFS OF THE RECTANGLE TILING PROBLEM

KARTHIK BALAKRISHNAN

Theorem - Whenever a rectangle is tiled by rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side.

Sperner's Lemma

Proof. Let us begin by placing rectangle R in standard position. Assume that the theorem is false and neither the width or the height of rectangle R is an integer value. Now draw a diagonal in every small tile, thereby triangulating R. Now, color a vertice green if its x-value is an integer, color a vertice red if only the y-value is an integer, and color a vertice blue if neither the x nor y value is an integer. By Sperner's Lemma, we should have an odd number of rainbow triangles (i.e. triangles that have an a vertice of 3 different colors) however, notice that if any vertice is either red or green, then one of the adjacent vertices must also be red or green respectively. Our hypothesis states that each small tile must have at least two red or two green points. Since two of the points that define a triangle are of the same color in any given smaller tile, there are exactly zero rainbow triangles. Thus, by contradiction, our initial assumption must be false and R must have either integer side length, ■

Polynomial

Proof. Place rectangle R in standard position. Assume the hypothesis is false and R has no integer side lengths. Take all of the segments of the smaller tiles that do not have an integer length and translate them x' or y' where x'=x+t and y'=y+t where t is some value in the range (0, n). All the vertical segments should be translated to the right and all the horizontal segments should be translated up. If t is sufficiently small then a new rectangle R' will be formed with the same number of tiles as R. The new rectangle will have an area of $(a + t) * (b + t)$ and the area has a quadratic relation to t. If the hypothesis is true then R' has an area of $w * (h + t)$, $(w + t) * h$, or $w * k$. Since t is bounded by a sufficiently small range, the quadratic increase in area does not work and the hypothesis must be true as the linear increase in area does work. ■

Eulerian Path

Proof. Let us draw a graph T, such that there is a segment connecting two adjacent points if the distance between them is an integer length. Each tile point that is not on the corner is either a vertice to 2 or 4 tiles, meaning that it will have 2 or 4 edges sprouting out of it which will be in T. Vertices on the corner's however will have only one edge in T. If one where to take a stroll down graph T, from (0,0) they would only stop when the graph terminates at another corner. This means that it is possible to go from one corner to the other in strictly integer lengths thereby proving the theorem. ■