The Sylvester-Gallai Theorem

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1 The Theorem

The statement of the Sylvester-Gallai theorem is fairly innocuous— it merely states that, given a finite set of at least three points P in the Euclidean plane $\mathbb{R}^2$ (not all collinear) there exists a line l passing through *exactly* two of those points. The theorem seems obvious, at least in the sense that it is hard to conceive what a counterexample might look like (even in a non-Euclidean geometry where counterexamples do exist), but despite this simplicity the theorem was only proven decades after it was first posed.

2 Proof Number One

In the following proof, pay attention to which properties of Euclidean space we use in this proof, as those properties will be important in determining to which kinds of spaces this theorem generalizes.

Proof: Let P be a finite set of points, not all collinear. Let L be the set of all lines passing through at least two points in P. Consider the set of pairs (p, l) where $p \in P$ and $l \in L$. Since P is finite, L is finite and there are finitely man such pairs. Hence, we can consider the pair (p_0, l_0) such that the distance $d(p_0, l_0)$ is minimal. Consider the point k on l_0 so that the line from p_0 to k is perpendicular to l_0. If the Sylvester-Gallai theorem were false, then l_0 passes through three points in P, so on one side of k there are two points in P. (If k itself is in P, then that counts as either side.) Let p_1 be the point nearer to k and let q_1 be the one further. Consider the line l_1 passing through p_0 and q_1. This line is in L, since it passes through at least two points in P. p_1 is in P by definition. So we have a pair (p_1, l_1) in $P \times L$, and it is clear from the image that the distance $d(p_1, l_1)$ is shorter than $d(p_0, l_0)$. However, we assumed that the latter distance was minimal, so we have a contradiction. Hence, we have proven the Sylvester-Gallai theorem.

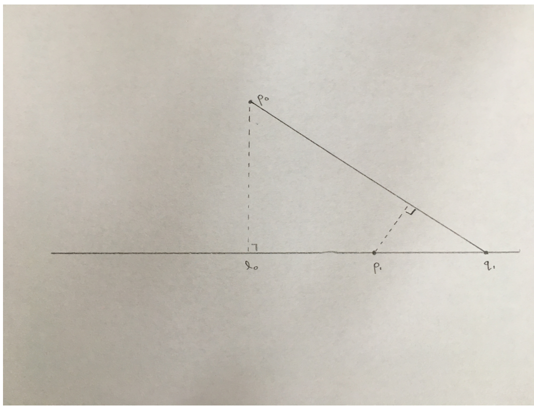

Figure 1: depiction of the first proof

Which properties of $\mathbb{R}^2$ did we use in that proof? First, we used the concept of distance when we declared $d(p_0, l_0)$ was minimal. Second, we used the concept of betweenness when we constructed p_1 and q_1. It turns out that betweenness is the only concept necessary for the proof, and in any space with a sufficiently strong notion of betweenness the Sylvester-Gallai Theorem will be true. One might first try to adapt this first proof into one that does not use distance, for example by constructing p_1 and q_1 as above and then analogously constructing a p_2 and q_2, etc. There would be infinitely many points in a set of finitely man points. However, it is not clear how to show that these points are all distinct without using some concept of distance. In order to prove the theorem without using distance, we must first introduce the concept of a 'projective dual.'

3 Projective Planes and Duality

A projective plane is a set of 'points' P and a set of 'lines' L, along with an incidence relation I that determines which points lie on which lines. I must satisfy the following properties:

1. Given two points, there is exactly one line passing through both points.
2. Given two lines, there is exactly one point lying on both lines.
3. Every line contains at least three points. (Note that this does not contradict the Sylvester-Gallai theorem because the set of points in the theorem do not form a projective plane.)
4. There exists a set of four points so that no line passes through more than two points of that set.

For an example, take the Fano plane: the points are the points, the lines are the lines and the circle, and the incidence relation is the obvious one where the points drawn on specific lines lie on those lines. It is easy to check that that relation satisfies the four properties. (For an example set for the last axiom, take each corner and the center.) Note that the Euclidean plane $\mathbb{R}^2$ is *not* a projective plane, because parallel lines do not intersect.

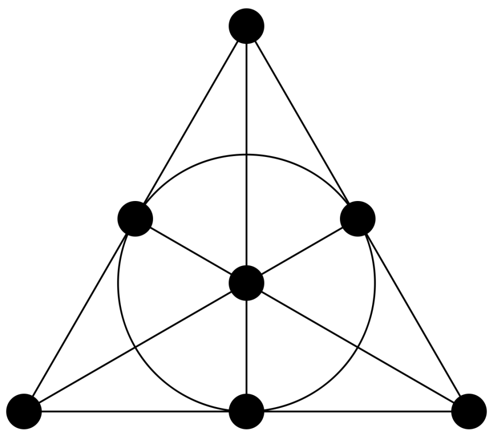

Figure 2: The Fano Plane (image due to Wikipedia)

The points and lines in a projective plane do not need to be literally points and lines: we saw in the Fano plane that a circle could be considered a line. In this example, the circle still seems somewhat 'linish,' as it is one dimensional in the diagram. However, in general P and L are just sets. There is no reason elements of L should resemble one-dimensional objects. A corollary of that realization is the concept of the dual plane.

Given a projective plane (P, L, I), we define its projective dual to be the plane (L, P, I^*), where l lies on p with respect to I^* if and only if p lies on l with respect to I. It is not difficult to check that I^* satisfies the above axioms. We can translate between theorems in a plane and its dual using the concept of plane dual statements. The plane dual of some statement simply replaces 'line' with 'point' and 'point' with 'line' and related substitutions. For example, if one wanted to say, "three points lie on the same line" the dual statement would be "three lines intersect at the same point." If a statement is provable in some plane, its dual must be provable in the dual plane, as the dual proof can be constructed by simply taking the dual of each line in the original proof. Often, the dual theorem is easier to prove than the original theorem, as is the case with the Sylvester-Gallai Theorem.

What is the dual of the Sylvester-Gallai Theorem? First, let us restate the original version of the theorem: "given a finite set of at least three **points** in $\mathbb{R}^2$, not all **on** the same **line**, there exists a **line passing through** exactly two **points** in that set." To dualize this statement, replace line with point and point with line: "given a finite set of at least three **lines**, not all **passing through** the same **point**, there exists a **point lying on** exactly two **lines** in that set." However, this dual statement is incomplete: in the original version, we know the points are in $\mathbb{R}^2$. However, in the dual version, the lines lie not in $\mathbb{R}^2$, but in the projective dual of $\mathbb{R}^2$. However, $\mathbb{R}^2$ is not a projective plane, so it doesn't have a dual. Fortunately, this problem can be fixed with the concept of the real projective plane.

4 The Real Projective Plane

The real projective plane $\mathbb{RP}^2$ is a generalization of the Euclidean plane so that any two lines intersect. We can define it by adding a 'point at infinity' for every possible slope m (there is also a point for vertical lines with no or infinite slope). All lines of slope m (and only lines of slope m) pass through the corresponding point at infinity. We define the line at infinity consisting of every point at infinity.

We can check that this plane satisfies each axiom. First, any two points define exactly one line: if both points are finite, the line is whichever Euclidean line passes through them; if one is finite and one is infinite (say corresponding to slope k), the line is the unique line with slope k passing through the finite point. Finally, two infinite points define the line at infinity.

Hence the first axiom holds. For the second axiom, we know that Euclidean lines with different slopes intersect wherever they would in the Euclidean plane, parallel lines intersect at the corresponding point at infinity, and the line at infinity intersects a Euclidean line at the point at infinity which corresponds to its slope. The third axiom is obvious from the construction of the plane. For the final axiom, we can take any four points defining a quadrilateral in Euclidean space, since the additional line at infinity will not affect those points' independence.

To get a visualization of the real projective plane, one can perform the following construction. Imagine a sphere placed above the Euclidean plane in 3-space. Given some point in the plane, consider the line passing through that point and the center of the sphere. This line intersects the sphere at a pair of antipodal points. Every point in the plane can be mapped to a pair of antipodal points of the sphere in this manner. Lines are mapped to great circles (or equators) of the sphere. At this point, every point on the sphere is covered except for the equator that is parallel to the plane. That is where we bring in the points at infinity. Each point at infinity corresponds to a slope. Each slope corresponds to a line passing through the unmapped equator and the center of the sphere. Thus, each point at infinity corresponds to a pair of antipodal points on the equator, and vice versa. So one can think of the real projective plane as a sphere with antipodal points identified.

Now that we have defined the real projective plane, we can discuss its dual. Recall that in a dual plane, the points become lines and the lines become points. What happens if we do this to the real projective plane? It turns out that $\mathbb{RP}^2$ is its own dual. We can prove this using the spherical model of the projective plane. Given a pair of antipodal points on the sphere, they define a line passing through the center of the sphere. There is a unique plane normal to that line, and the intersection of the plane with the sphere defines a great circle— that is a line in the real projective plane. Working backwards, a great circle defines a plane, and there is a unique line normal to that plane passing through the center of the sphere. The intersection of that line with the sphere is a pair of antipodal points, i.e. a point in the real projective plane. This mapping is clearly a bijection, to check that it is an isomorphism we must show that it preserves incidence. If some point p lies on a great circle, then the line passing through p and the center of the sphere lies within the plane defined by the circle. Thus, the line normal to that plane lies within the plane normal to the line generated by p. Similarly, the intersection of that line with the sphere lies on the intersection of the plane with the sphere. Hence incidence is preserved and it is an isomorphism.

5 Proof Number Two

Now that we have the vocabulary of the real projective plane, we can finally prove the Sylvester-Gallai theorem without using distance. First, instead of points in $\mathbb{R}^2$, we will prove the theorem for points in $\mathbb{RP}^2$. Any counterexample in the Euclidean plane gives rise to a counterexample in the real projective plane, since the line at infinity is the only additional line, and it will not pass through any points in the Euclidean plane. So proving the theorem for the projective plane will imply the result in the Euclidean plane.

Since $\mathbb{RP}^2$ is its own dual, we can prove the dual of the Sylvester-Gallai theorem instead. That is, we will prove the following theorem:

Theorem: Given a finite set at least three lines in the real projective plane, not all passing through a single point, there exists a point lying on exactly two lines in that set.

Proof: Since not all of the points intersect at the same point, pick three that don't. They partition the projective plane into four distinct triangular regions. Below is a diagram illustrating these four regions. Now consider the point Q_0, the intersection of two of those lines. If Q_0 lies on only two lines, we are done, so assume there is a third line l_0. Pick one of the triangular regions into which that line passes. It partitions that triangle into two smaller triangles, and it intersects the base of the triangle at some point. Call that point Q_1. Again, if Q_1 lies on just two lines we are done, so assume there is a third line l_1. l_1 passes through one of the smaller triangles cut by l_0. Repeating the process, Q_2 is the intersection with l_1 and the base of that triangle. *We know that Q_2 is distinct from the previous points because it lies within a triangle in which the other points do not lie.* Similarly, we can construct Q_3, Q_4, etc., and since each of these points exists in a new smaller triangular region they must be distinct. However, a finite set of lines cannot have infinitely many intersections, so we have a contradiction.

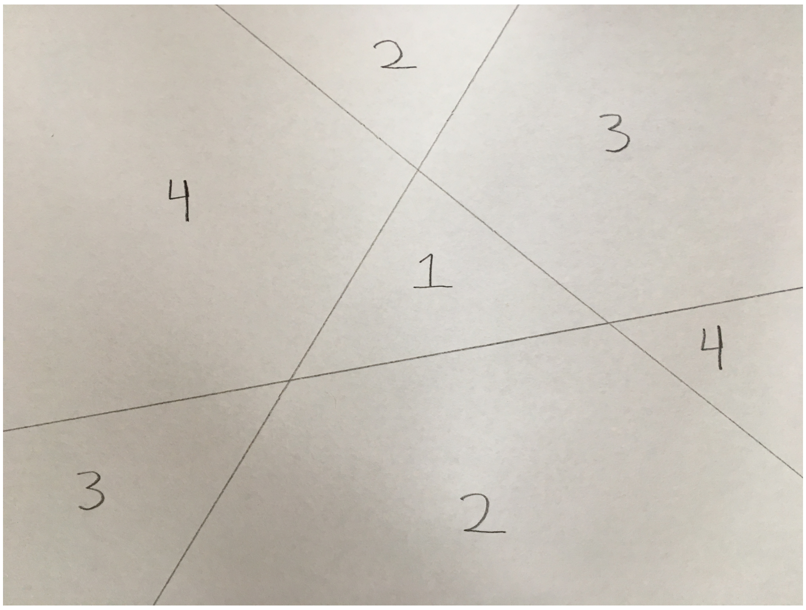

Figure 3: The numbered regions are the four triangular regions in the projective plane.

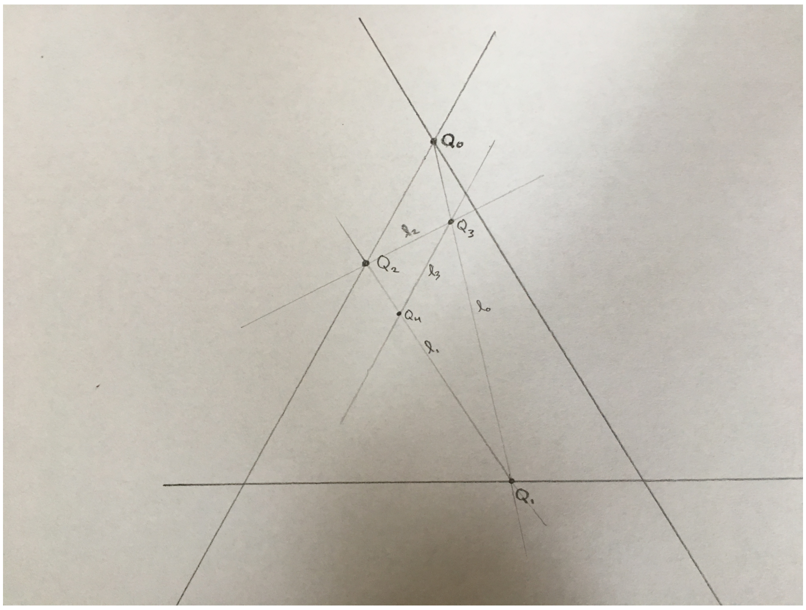

Figure 4: depiction of the second proof

6 Resources

Proofs from the Book, Fifth Edition by Martin Aigner

https://math.berkeley.edu/~monks/papers/DualityV3.pdf

http://kahrstrom.com/mathematics/documents/OnProjectivePlanes.pdf

http://www.research.ibm.com/people/l/lenchner/docs/sylvester_note_geom.pdf

http://web.science.mq.edu.au/~chris/geometry/CHAP01%20The%20Real%20Projective%20Plane.pdf