ART GALLEY THEOREM AND OTHERS

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1. The Art Gallery Problem

The original art gallery problem is as follows: given a polygonal art gallery, what is the fewest number of guards that need to be placed in the gallery such that the guards can see the entire gallery?

This problem statement can be made more rigorous with a few definitions:

Definition 1.1. A *guard* is a single point located in the interior of the art gallery.

Definition 1.2. A point is 'visible' to a guard if the segment between the guard and the point does not cross the boundary of the polygon.

Definition 1.3. A set of guards *covers* a gallery if every point in the interior and boundary of the gallery is visible to at least one guard.

The question then becomes: What is the fewest number of guards required to cover a polygonal art gallery.

2. Chvátal's Theorem

Chvátal's Art Gallery Theorem, sometimes referred to as 'the' Art Gallery Theorem, was proved by Václav Chvátal in 1973 and states that $\lfloor \frac{n}{3} \rfloor$ guards are sometimes necessary and always sufficient to cover an n-gon. The necessity case can be proven using a comb-shaped figure, a trapezoid adjoined with many triangles. Chvátal's original proof of sufficiency was much more complicated, however, it was soon simplified by Steve Fisk.

3. Fisk's Proof

Steve Fisk's proof is as follows: First, triangulate the n-sided polygon using only internal diagonals, which, by the Triangulation Theorem, is possible and yields a set T of triangles with cardinality $n-2$. Now, define graph G with the vertices of the polygon as nodes, where 2 nodes are connected if the connected by a side of the polygon or by a diagonal that is part of the triangulation. Define graph G_T with the elements of T as nodes. Nodes of G_T are connected if the triangles share an edge. If there was a cycle in G_T, then there would be a hole inside the figure. However, the figure is a polygon, which does not have any holes, thus G_T has no cycles and is a tree. Because of this, we can now take some element T_0 of T and color its vertices with three different colors in G. From there, G can then be 3-colored by taking an element of T adjacent to T_0 and assigning the vertex not shared with T_0 to the color not shared with T_0, and continuing to do this for adjacent nodes of G_T until all of G has been 3-colored. Because G_T is a tree, each node of G_T has only one path from T_0. Once G is completely 3-colored, take any of the 3 colors. Because all triangles are convex, a guard placed at any vertex of a triangle completely covers it. Therefore, the set of all vertices of

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one color cover every triangle in the polygon, and thus the entire polygon. Because this holds for all colors, we can select the color with the fewest number of vertices, which would be at most $\lfloor \frac{n}{3} \rfloor$, proving sufficiency.

4. Orthogonal Galleries

The above proofs have looked at polygonal galleries, however, what happens when we restrict ourselves to orthogonal polygons? Using a similar comb structure, we find that it is sometimes necessary to use $\lfloor \frac{n}{4} \rfloor$ guards to cover this polynomial. The sufficiency of t condition is much harder to prove, as it requires proving the existence of a convex quadrangulation Q, which is not nearly as trivial as a triangulation is. However, by defining 'neighboring edges' as the closest pair of opposite edges in view of each other, and using this definition to define 'tabs' which stick out of the rest of the polygon, a valid convex quadrangulation can be constructed. With this quadrangulation, a 4-coloring can be made such that each quadrangle contains one vertex of each color. From here, the same proof as the general case applies, but the number of vertices of the least-occurring color is at most $\lfloor \frac{n}{4} \rfloor$, proving sufficiency.

5. Galleries with holes

Another extension of this problem deals with galleries that contain obstructions, or 'holes' in the polygon. In 1991, Hoffmann, Kaufmann, and Kreigel proved that that $\lfloor \frac{n+h}{3} \rfloor$ guards were both necessary and sufficient, where h is the number of obstructions/holes and n is the number of sides of the gallery, including the holes. This proof is somewhat more complex, as the graph G_T in Fisk's proof is not necessarily a tree.

For orthogonal galleries with obstructions, a somewhat similar proof to both the proof for the orthogonal polygons and the proof for polygons with holes yields a result of $\lfloor \frac{n+h}{4} \rfloor$ guards being sometimes necessary and always sufficient. This result is very much in line with the three previous cases.

6. The Fortress Problem

While the Art Gallery Problem asks for a set of guards that can cover the entire interior of a polygon, the Fortress Problem asks for a set of guards on vertices of the polygon that can cover the entire exterior of the polygon. To do this, take the convex hull C of the polygon P. Then, we define a 'point at infinity', or a point P_0 extremely far away from the polygon. We then connect this point to every point on C. Some of these connections will not be straight lines. Then, take some vertex on C and split it into two vertices. Now, the entire exterior of the polygon can be opened up to form another polygon with $n+2$ vertices: n vertices of the polygon, as well as the duplicate vertex and the point at infinity. This polygon can then also be triangulated by connecting P_0 to every pair of adjacent vertices of C. Then, each of the 'cavities' inside the convex hull can be triangulated separately. Once this triangulation has been generated, a 3-coloring can be generated by the same method used in the Art Gallery Theorem. Once again, we may pick the a set of points of one color, however, one of these sets includes P_0, and this set S_0 cannot be picked. However, if we pick the set of another color (call it S), then S has a vertex other than P_0 on each element of the triangulation T. This means it has one vertex on each edge of the convex hull, and for convex polygons, having one guard on each edge covers the entire exterior, thus, S as the set of guards covers the entire exterior of C as well as all the triangles inside C and outside P, thus, it covers the entire exterior of P. We can now pick the smaller of the two sets S_1 and S_2. Because S_0 contains at least one vertex, P_0, there are at most $n+1$ vertices to be split between the two colors, thus, the maximum size of the smaller set is $\lfloor \frac{n+1}{2} \rfloor$, or $\lceil \frac{n}{2} \rceil$. In this case, any convex polygon can prove necessity.

For the exterior of orthogonal polygons, a large rectangle can be drawn outside the polygon with an extra rectangle K being cut off to connect the rectangle with the topmost edge of the polygon, creating a new polygon whose interior is the area inside the rectangle but outside the polygon and K. From here, the proof relies on a slightly different theorem related to the number of reflex vertices of the polygon, which gives that $\lfloor \frac{n+4}{4} \rfloor$ guards on reflex vertices are sufficient to cover this region. As none of the newly created vertices are reflex vertices, this guard set covers the entire exterior except for K, which can be covered by adding one more guard, giving a total maximum of $\lfloor \frac{n}{4} + 2 \rfloor$ vertex guards. Necessity can be proven using a simple orthogonal sprial structure.

Remark 6.1. Note that both these proofs require guards to be located on the boundary of the polygon. If guards were allowed to be placed outside the polygon, the required number of guards would decrease, especially for the non-orthogonal case, as the exterior of the convex hull could be covered by two guards far enough away.

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