Monsky's Theorem

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Abstract

We examine Monsky's theorem, which states that if we try to dissect a unit square into n triangles of equal area, then n must be even. We prove this through applications of Sperner's Lemma in combinatorial topology and 2-adic valuations.

1 Introduction

Consider the following situation: we want to dissect a square into n nonoverlapping triangles of equal area. This is trivial if n is even: we can divide the horizontal sides of the square into $\frac{n}{2}$ segments of equal length, which determines a dissection of the square into $\frac{n}{2}$ congruent rectangles and then draw the diagonal of each rectangle (see Figure 1). What about the case when n is odd? Is such a dissection possible?

This problem arose when Fred Richman was preparing a master's exam, in 1965. In fact, the question was first asked about a rectangle, instead of a square. If we consider the cartesian coordinate plane (the xy-plane), using horizontal or vertical dilations, if necessary, we can assume without loss of generality that our rectangle is the unit square.

At first the problem seemed to be easy, but Richman could not find a complete solution for it. He found the such a dissection is impossible for $n = 3$ and 5, and if it exists for some n, then it exists for $n+2$. He presented the question to his colleague, John Thomas, who got some more progress toward the solution, but could not solve it for completely. Thomas found that for the unit square there is no such a dissection where all the vertices of the triangles have rational coordinates with odd denominators.

We present the proof of Monsky's theorem based on applications of 2-adic valuations, a combinatorial topology result (Sperner's Lemma), and the fact that those can be extended to $\mathbf{R}$. We will end with further applications such as balanced polygons in tropical geometry and similar theorems inspired by Monsky's theorem.

2 Background

2-adic Valuations

Definition 1.1: Consider a field F. The *absolute value* $|\cdot|$ on F is a map to $\mathbb{R}$ and satisfies the following properties:

1. $|x| > 0$ for all $x \in F \neq 0$
2. $|x| = 0$ if and only if $x = 0$
3. $|xy| = |x||y|$ for all $x, y \in F$
4. $|x + y| \leq |x| + |y|$ for all $x, y \in F$

If it also satisfies the ultrametric inequality $|x + y| \leq max(|x|, |y|)$ for all $x, y \in F$ then the absolute value is called *non-archimedean.*
Definition 1.2: The 2-adic non-archimedean absolute value on $\mathbb{Q}$ is $|\cdot|_2 : \mathbb{Q} \to \mathbb{R}$ defined by $|x|_2 := 2^{-v_2(x)}$, where $v_2(x) = v_2(2^n \frac{a}{b})$ where a, b are odd integers and $n \in \mathbb{Z}$.
Lemma 1: Claude Chevalley's theorem states that there exists such a function which extends the p-adic absolute value to $\mathbb{R}$, and we will denote this function as $|x|'_p = p^{-v_p(x)}$
Proof. (do we need to prove this? Will add later if needed)

Sperner's Lemma

First, let a labelling of the plane into 3 labels, P_0, P_1, P_2 be determined by the following:

- $P_0 := (x, y) : |x|'_2 < 1, |y|'_2 < 1$
- $P_1 := (x, y) : |x|'_2 \geq 1, |x|'_2 \geq |y|'_2$
- $P_2 := (x, y) : |y|'_2 \geq 1, |y|'_2 > |x|'_2$

Definition 2.1: Given a polygon P, the *dissection* of P is the partitioning of the polygon into triangles. *Equidissection* is the dissection of P into equal parts.
Definition 2.2: Consider a polygon P that is dissected into triangles T_i. Label the vertices of the triangles P_0, P_1, and P_2. A segment is called a *complete edge* if its endpoints are P_0 and P_1. If the triangle has corner endpoints of all three vertices, we will call the triangle a *complete triangle*.

Sperner's Lemma: Consider a triangulation of a polygon R such that each vertex is labeled either P_0, P_1, or P_2. Then the number of complete triangles in R and the number of complete edges on the boundary of R have the same parity.
Proof. We apply a double counting combinatorial argument. Place a dot on each side of a complete edge. We want to count the number of dots in two

different ways. Firstly, each interior segment contributes either 0 or 2 dots, while each boundary segment contributes either 0 or 1 dots. Next, we count the number of dots in the interior of each triangle in the dissection. By construction, complete triangles contain one dot while the rest contain an even number of dots. Therefore, the parity of the number of the dots in the interior of all the triangles is equal to the number of dots contribute by all the contributed by all the complete edges on the boundary. Essentially, the number of dots is equal to the number of complete edges on the boundary of R (mod 2). Therefore, the parity of the number of complete edges on the boundary of R is the same as the number of complete triangles in R.

[Sperner's lemma can be used to relate the boundary of the square to the triangles in its dissection. However, in order to apply this lemma, we will need a method of labelling the vertices of the triangulation. Conveniently, the p-adic absolute value can serve this purpose.]

Proposition 2.3 Let points (x_0, y_0), (x_1, y_1), and (x_2, y_2) be points with labels P_0, P_1, and P_2 respectively. Then, we claim that shifting (x_1, y_1) and (x_2, y_2) relative to (x_0, y_0) or by subtracting x_0 from the x coordinate and subtracting y_0 from the y coordinate will not alter their labels.

Proof. If we take (x_1, y_1) and subtract by x_0 and y_0, then we get the point $(x_1 - x_0, y_1 - y_0)$. Then, we need to show that this new point also has the label P_1. In order to do so, we can just consider $|x_1 - x_0|_2'$. We know that $|x_0|_2' < 1$, and that $|x_1|_2' \geq 1$, thus because the p-adic absolute value is non-archimedean, we know that $|-x_0|_2' = |x_0|_2' < |x_1|_2'$, so we can conclude that $|x_1 - x_0|_2' = |x_1|_2' \geq 1$. Similarly, we can conclude that $|y_1 - y_0|_2' = |y_1|_2'$ or $|y_1 - y_0|_2' = |y_0|_2'$, but $|y_0|_2' < 1 \leq |x_1|_2'$ and $|y_1|_2' \leq |x_1|_2'$, thus $|y_1 - y_0|_2' \leq |x_1 - x_0|_2'$ so the point $(x_1 - x_0, y_1 - y_0)$ has the label P_1.

Now, we can do the same to show that if we take the point (x_2, y_2) and subtract it by x_0 and y_0, then the resulting point $(x_2 - x_0, y_2 - y_0)$ must also be in P_2. $|y_2|_2' \geq 1 > |y_0|_2' = |-y_0|_2'$. Therefore by the non-archimedean property, we must have $|y_2 - y_0|_2' = |y_2|_2' \geq 1$. Similarly, we must have $|x_2 - x_0|_2' = |x_2|_2'$ or $x_2 - x_0|_2' = |-x_0|_2' = |x_0|_2'$. However, we have that $|x_0|_2' < 1 \leq |y_2|_2'$ and $|x_2|_2' < |y_2|_2'$, thus we have $|x_2 - x_0|_2' < |y_2 - y_0|_2'$, so the point $(x_2 - x_0, y_2 - y_0)$ has the label P_2.

Therefore, we have proven Proposition 2.3 and we now know that we can translate a complete triangle by its P_0 point, which shifts the triangle such that the P_0 point goes onto the origin, and we know that the remaining two points still have the same label.

Now, we wish to find the area of some complete triangle.

Lemma 2.4 Let area of a complete triangle be S. Then we claim that

$$|S|_2' > 1$$

Proof. If this triangles has vertices with the labels of P_0, P_1, and P_2 in some order, then we first shift the triangle such that the point with label P_0 becomes the origin using Proposition 2.3. Now, the other two points still have labels P_1

and P_2 according to Proposition 1. Let the point with label P_1 have coordinates (x_1, y_1) and the point with label P_2 have coordinates (x_2, y_2). Then, by the Shoelace Theorem, we can find the area of this triangle to be

$$S = \frac{|x_1y_2 - x_2y_1|}{2}$$

First notice that whether or not this is positive or negative does not affect the 2-adic absolute value of S, so we can first ignore the absolute value. Then, if we consider this value 2-adically, we get the following:

$$|S|_2' = |\frac{x_1y_2 - x_2y_1}{2}|_2'$$

$$|\frac{x_1y_2 - x_2y_1}{2}|_2' = |\frac{1}{2}|_2' \cdot |x_1y_2 - x_2y_1|_2'$$

Now, we can compare the values of $|x_1y_2|_2'$ and $|x_2y_1|_2'$. By the definition of the labelling, we have that $|x_1|_2' \geq |y_1|_2'$, as well as $|y_2|_2' > |x_2|_2'$. Therefore, we know that $|x_1y_2|_2' = |x_1|_2' \cdot |y_2|_2' > |y_1|_2' \cdot |x_2|_2' = |x_2y_1|_2' = |-x_2y_1|_2'$. Thus, by the non-archimedean property of 2-adic absolute value, we know that $|x_1y_2 - x_2y_1|_2' = |x_1y_2|_2'$. Moreover, by the definition of the labeling we also know that $|x_1|_2' \geq 1$ and $|y_2|_2' \geq 1$, thus $|x_1y_2|_2' \geq 1$. Therefore, $|S|_2' = |\frac{x_1y_2 - x_2y_1}{2}|_2' = |\frac{1}{2}|_2' \cdot |x_1y_2 - x_2y_1|_2' \geq |\frac{1}{2}|_2' = 2 > 1$.

In conclusion, we know that the area of a complete triangle must be at least 2 (or greater than 1), 2-adically.

Now, we can prove Monsky's Theorem.

3 Monsky's Theorem

Proof. First, for any square in the plane, we can translate and dilate it such that it is a unit square, and has vertices at points $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. Then, after shifting we can label each point in R^2 according to the 2-adic absolute value of its coordinates according to the following labelling:

- $P_0 := (x, y) : |x|_2' < 1, |y|_2' < 1$
- $P_1 := (x, y) : |x|_2' \geq 1, |x|_2' \geq |y|_2'$
- $P_2 := (x, y) : |y|_2' \geq 1, |y|_2' > |x|_2'$

(the same labelling is the same labelling we used earlier in the paper.)

Now, note that $v_2(0) = \infty$, therefore $|0|_2' = p^{-\infty} = 0$. Thus, the point $(0,0)$ will have label P_0, point $(1,0)$ will have label P_1, point $(0,1)$ will have label P_2, point $(1,1)$ will have label P_1.

Now, we can consider where the complete edges on the boundary are. As complete edges are edges with endpoints P_0 and P_1, we can disregard points on the edge from $(0,0)$ to $(0,1)$, as the x coordinate is always 0, therefore the

2-adic labelling will never include points P_1. Similarly, the edge from $(1,0)$ to $(1,1)$ will never contain points with label P_0 as the x coordinate is always 1, which is exactly 1 in the 2-adic absolute value, and in order for it to be labelled P_0, $|x|_2' < 1$, $|x|_2' \neq 1$. Finally, the edge from $(0,1)$ to $(1,1)$ will never contain points with label P_0 either since its y-coordinate is always 1, meaning that the 2-adic absolute value will be exactly 1.

Therefore, we know that the only possible complete edges on the boundary must exist only from $(0,0)$ to $(1,0)$. Moreover, we claim that there is an odd number of complete edges. First, we can show that there will never be a P_2 point on the edge from $(0,0)$ to $(1,0)$. Clearly this is true as the y-coordinate is always 0, and $|0|_2' = 0 < 1$. Then, we know that on the edge from $(0,0)$ to $(1,0)$ there are only P_0 and P_1 points. Now, we can prove that there is an odd number of complete edges by strong induction.

Lemma 3.1 Consider n points on the edge from $(0,0)$ to $(1,0)$ including its endpoints, such that $n \geq 2$ and each point is either a P_0 point or a P_1 point. We claim that there is an odd number of complete edges.
Proof. The base case is where there is only two points, in our case the two vertices $(0,0)$ and $(1,0)$. This clearly has an odd number of complete edges. Now, consider n points on this edge. If we assume that the statement is true for 2 through $n-1$ points, and they all have an odd number of complete edges, then we can show that n points must also have an odd number of complete edges. As we are looking at $n \geq 3$ (not the base case), we can consider some point X in the middle of the edge (not $(0,0)$ or $(1,0)$). Then, this point could either be a P_0 or a P_1 point. First, let us deal with the case when X is a P_0 point. Now, we consider the two points closest to X on each side (one closest on the left, one closest one the right), and then consider what happens to the number of complete edges when we include or exclude X.

- Case 1: Both points have label P_0. In this case, without adding the point X the two adjacent P_0 points contributed 0 complete edges to the total. When we add back the point X, There are 3 P_0 points, which still contributes 0 complete edges. Therefore the parity remains the same.
- Case 2: Both points have label P_1. In this case, when we consider the point X in between, there are 2 complete edges. However, when we remove X, there is two adjacent P_1 points, so there are 0 complete edges. Therefore, in this case the parity of the number of complete edges remains the same.
- Case 3: One point has label P_1 and the other has label P_0. In this case, when we consider the point X in the middle, it will split the segment into a P_1P_1 and a P_1P_0 (or P_0P_1), therefore X contributes one complete edge. However, when we remove X, there is also exactly one complete edge created by the two points surrounding X, so the parity of the number of complete edges remains the same.

Notice that when counting the parity, we only need to consider the two points surrounding X as other points will not be relevant to a complete edge as only adjacent points have edges connecting them on the boundary.

Now, we can repeat the same casework except we consider when X is a P_1 point this time. Similar to above, the parity of complete edges never changes. Therefore, the parity is the same as the base case, which is odd.

Now, by Sperner's lemma, we know that the number of complete triangles inside of the dissection of the square must also be odd, which implies that there must at least be one complete triangle in the dissection.

However, by Lemma 2.4, we know that the 2-adic absolute value of the area of a complete triangle must be greater than 1. Therefore, if we let the area of each triangle in the equal dissection of the square be s, and the number of triangles be m, then $ms = 1$. Now, if we consider this with the 2-adic absolute value, we know that

$$|ms|_2' = |1|_2' = 1$$

$$|m|_2' \cdot |s|_2' = 1$$

Lemma 2.4 states that $|s|_2' > 1$, therefore $|m|_2' < 1$ as the product is 1 and the 2-adic absolute value is multiplicative. As m is an integer (the number of triangles in the dissection), and its 2-adic absolute value is less than 1, this means that m must be even. Therefore, we have proved Monsky's Theorem, that any dissection of a square into triangles of equal area must have an even number of triangles.

4 Applications

Monsky's theorem can be generalized to the study of balanced polygons, a key topic in tropical geometry, a variant of algebraic geometry.

Definition 3.1: Let B be a plane polygon with clockwise oriented boundary.B is called *balanced* if its edges can be divided into pairs so that in each pair, edges are parallel equal in length, and have opposite orientation (the edges are oriented, there orientation comes from the orientation of the boundary)

This introduces the Stein conjecture.

Corollary (S.Stein, 2000). A balanced polygon cannot be cut into an odd number of triangles of equal areas.

Theorem 4.1 *(Non-equidessectibility of a balanced lattice polygon). Consider a balanced polygon B of the integer odd area and assume that the coordinates of all the vertices are integer numbers. Then B cannot be cut into an odd number of triangles of equal area.*

Monsky's theorem has inspired similar theorems to be made which also are generalizations of the p-adic absolute value and Sperner's Lemma.

1. Partitioning the n-dimensional cube into simplices yields that the number of simplices must be a multiple of $n!$.
2. Partitioning regular n-gons for $n > 4$ implies that the number of triangles is divisible by n.
3. Monsky also showed that for a centrally symmetric polygon, the answer is the same as that for a square.
4. There are some polygons that cannot be dissected into triangles of equal areas. An example is the trapezoid with vertices $(0,0),(0,1),(1,0),(a,1)$, where a is not algebraic.

References

[1] William Sablan (2019) *An Exposition of Monsky's Theorem*, University of Chicago

[2] Henry Swanson (2018) *Monsky's Theorem*, Math Mondays, `https://mathmondays.com/monskys-theorem`.

[3] Andreu Ferre Moragues (2016) *What is... Monsky's Theorem?*, University of Ohio