THE p-ADIC SOLENOID

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Contents

Abstract. The fields $\mathbb{R}$ of real numbers and $\mathbb{Q}_p$ of p-adic numbers can be linked in an interesting topological group, the solenoid. In this paper we will descirbe properites.

1. Introduction

Firstly, we give some essential definitions such as projective system, abelian group, isomorphism and the p-adic Solenoid.

Later, we will describe some theorems and their corollary. Also we will talk about p-adic integers.

2. Definitions

Definition 2.1. The p-adic Solenoid

For $p$prime and $n \in \mathbb{Z}_{\geq 0}, p^n\mathbb{Z}$ is a closed subgroup of the locally compact $n \geq m$, let $\phi_{n,m} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z}$ be the projection map, which is a morphism. The compact abelian groups $\mathbb{R}/p^n\mathbb{Z}$ and the morphisms $\phi_{n,m}$, are an inverse system, and the inverse limit is a compact abelian group denoted $\mathbb{T}_p$, called the p-adic Solenoid.

Definition 2.2. Projective system

A sequence $(E_n, \phi_n)_{\geq 0}$ of sets and maps $\phi_n : E_{n+1} \to E_n (n \geq 0)$ is called a projective system.

Definition 2.3. An *abelian group* is a group $(S, *)$ such that $*$ is also commutative; for all $a, b \in S$,

$$a * b = b * a.$$

Definition 2.4. A topological space X is *connected* if there cannot exist nonempty open subsets Y and Z of X such that $Y \cup Z = X$ and $Y \cap Z = \emptyset$.

Definition 2.5. Let $(A, *)$ and $(B, *')$ be two algebraic structures with binary operations. A *homomorphism* is a mapping $\psi : A \to B$ such that

$$\psi(x * y) = \psi(x) *' \psi(y).$$

If this mapping is bijective, it is called an *isomorphism.*

3. Properties

Theorem 3.1. *The p-adic solenoid contains a dense subgroup isomorphic to $\mathbb{R}$. It also contains a dense subgroup isomorphic to $\mathbb{Q}_p$.*

Proof. The projection maps $f_n : \mathbb{R} \to \mathbb{R}/p^n\mathbb{Z}$ are compatible with the transition maps of the projective system defining the solenoid

$$f_n = \phi_n \cdot f_{n+1} : \mathbb{R} \to \mathbb{R}/p^{n+1}\mathbb{Z} \to \mathbb{R}/p^n\mathbb{Z}.$$

Hence there is a unique factorization such that
$f_n = \psi \cdot f : \mathbb{R} \to \mathbb{S}_p \to \mathbb{R}/p^n\mathbb{Z}$
If $x \neq 0 \in \mathbb{R}$, as soon as $p^n > x$ we have $f_n(x) \neq 0 \in \mathbb{R}/p^n\mathbb{Z}$ and consequently $f(x) \neq 0 \in \mathbb{S}_p$. This shows that the homomorphism f is injective. The density of the image of f follows from the density of the images of the f_n. Consider now the subgoups

$$\mathbb{H}_k = \psi^{-1}(p^{-k}\mathbb{Z}/\mathbb{Z}) \in \mathbb{S}_p(k \geq 0)$$

We have $H_0 = \mathbb{Z}_p$ by definition, and this is a subgroup of index p^k of H_k :

$$H_k = \lim_{n\to} p^{-k}\mathbb{Z}/p^n\mathbb{Z} = p^{-k}\mathbb{Z}_p(k \geq 1)$$

Hence

$$\mathbb{Q}_p = \psi^{-1}(\mathbb{Z}[1/p]\mathbb{Z}) = U\psi^{-1}(p^{-k}\mathbb{Z}/\mathbb{Z}) = UH_k \in \mathbb{S}_p.$$

The density of this subgroup of $\mathbb{S}_p$ follows from the density of all images

$$\psi_n(\mathbb{Q}_p) = \mathbb{Z}[1/p]/p^n\mathbb{Z} \in \mathbb{R}/p^n\mathbb{Z}$$

■

Corollary 3.2. *The solenoid is a (compact and)* connected space.

Proof. Recall that for any subspace A of a topological space X we have

$$A \text{ connected, } A \in B \in \overline{A} \to B \text{ conneceted.}$$

In our contetxt, take for A the connected subspace $f(\mathbb{R}) \in \mathbb{S}_p$, which is dense in the solenoid. The conclusion follows.

Let us summarize the various homomorphisms connected to the solenoid in a communtative diagram.

$$\begin{array}{ccccc} \mathbf{Z} & \hookrightarrow & \mathbf{Z}_p & = & \mathbf{Z}_p \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{R} & \hookrightarrow & \mathbf{S}_p & \leftarrow & \mathbf{Q}_p \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{R}/\mathbf{Z} & = & \mathbf{R}/\mathbf{Z} & \leftarrow & \mathbf{Q}_p/\mathbf{Z}_p \end{array}$$

■

Theorem 3.3. *Every element of S_p can be uniquely formed by an element of $\mathbb{Z}_p$and a real number on the interval* $[0, 1)$.

Proof. First, note that there exists a subset of S_p isomorphic to $\mathbb{Z}_p$. This is because S_p has a surjective homomorphism to $\mathbb{R}/p^n\mathbb{Z}$ for $n \geq 0$. It follows that S_p has a surjective mapping to $\mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 1$, since for each n, $\mathbb{Z}/p^n\mathbb{Z} \in \mathbb{R}/p^n\mathbb{Z}$. Hence, we can represent any element of Z_p by mapping to an integer in $\mathbb{R}/p^n\mathbb{Z}$ for $n \geq 1$ and to 0 in $\mathbb{R}/\mathbb{Z}$. Now, take any $z \in S_p$. Suppose $z \equiv r \pmod 1$, where we take $0 \leq r < 1$. Then $z - r \equiv 0 \pmod 1$, and as a result $z - r \in Z_p$. Thus we may write

$$z = x + r,$$

where $x \in \mathbb{Z}_p$ and $r \in [0, 1)$. The proof of uniqueness is straightforward: let

$$z = x + r = x' + r'$$

where both $x' = x$ and $r' = r$. Since $r \neq r'$, then $r - r' \neq 0 \pmod 1$, a contradiction. So in order for $z = x+r$, to hold, we must have $r = r'$, and in turn, this forces $x = x'$. ■

Theorem 3.4. *The sum homomorphism $f : \mathbb{R} \cdot \mathbb{Q}_p \to \mathbb{S}_p$ furnishes an isomorphism $f' : (\mathbb{R} \cdot \mathbb{Q}_p)/\Gamma_p \equiv S_p$ both algebraically and topologically.*

Proof. Since all maps f_n are surjective, the map f has a dense image. Moreover, using the integral and fractional parts introduced there,

$$f(t, x) = f(t + (x), x - (x)) = f(s, y),$$

where $s \in \mathbb{R}$ and $y = x - x = [x] \in \mathbb{Z}_p$.Going one step further,we have

$$f(s,y) = f(s-[S], y+[s] = f(u,Z),$$

where $u = s - [s] \in [0, I)$ and $z = y + [s] \in \mathbb{Z}_p$.This proves

$$\operatorname{Im} f = f(\mathbb{R} \cdot \mathbb{Q}_p) = f([O,1)x\mathbb{Z}_p).$$

A fortiori, the image of f is equal to $f([O,1]x\mathbb{Z}_p)$, and hence is compact and closed. Consequently, f is surjective (and f' is bijective). In fact, the preceding equalities also show that the Hausdorff quotient (recall that the subgroup $\Gamma' p$ is discrete and closed) is also the image of the compact set $\Omega = [0, I]x\mathbb{Z}_p$ and hence is compact. The continuous bijection

$$f' : (\mathbb{R} \cdot \mathbb{Q}_p)/\Gamma_p \to \mathbb{S}_p$$

between two compact spaces is automatically a homeomorphism. ■

Corollary 3.5. *The solenoid can also be viewed as a quotient of* $\mathbb{R} \cdot \mathbb{Z}_p$ *by the discrete subgroup* $\triangle_z = (m, -m) : m \in \mathbb{Z}$

$$f' = (\mathbb{R} \cdot \mathbb{Z}_p)/\triangle_z = \mathbb{S}_p.$$

Proof. Since the restriction ofthe sum homomorphism $f : \mathbb{R} \cdot \mathbb{Q}_p \to \mathbb{S}_p$ to the subgroup $\mathbb{R} \cdot \mathbb{Z}_p$ is already surjective, this restrietion gives a (topological and algebraic) isomorphism

$$f' = (\mathbb{R} \cdot \mathbb{Z}_p)/\ker f' = \mathbb{Z}_p.$$

But $\ker f' = (\ker f) \cap (\mathbb{R}x\mathbb{Z}_p) = \triangle_z = (m, -m) : m \in \mathbb{Z}$. ■

4. p-adic integers

For $n \geq m$, let $\phi_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ be the projection map. With the discrete topology, $\mathbb{Z}/p^n\mathbb{Z}$ is a compact abelian group, as it is finite. Then $\phi_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ is an inverse system, and its inverse limit is a compact abelian group denoted $\mathbb{Z}_p$, called the p-adic integers, with morphisms ϕ_n: $\mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$. Because the morphisms $\phi_{n,m}$ are surjective, the morphisms ϕ_n are surjective.

Let $\lambda_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{R}/p^n\mathbb{Z}$ be the inclusion map. Then the morphisms $\Lambda_n = \lambda_n \cdot \phi_n : \mathbb{Z}_p \to \mathbb{R}/p^n\mathbb{Z}$ are compatible with the inverse system $\phi_{n,m}$:

$\mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z}$, so there is a unique morphism $\Lambda : Z_p \to T_p$ such that $\phi_n \cdot \Lambda = \Lambda_n$ for all $n \in \mathbb{Z} \geq 0$. Suppose that $x, y \in \mathbb{Z}_p$ are distinct and that $\Lambda(x) = \Lambda(y)$. It is a fact that there is some n such that $\phi_n(x) \neq \phi_n(y)$. Because λ_n is injective, this implies that $\Lambda_n(x) \neq \Lambda_n(y)$, and this contradicts that $\Lambda(x) = \Lambda(y)$. Therefore $\Lambda : \mathbb{Z}_p \to \mathbb{T}_p$ is injective. One proves that $\ker \phi_0 = \Lambda(\mathbb{Z}_p)$, so that

$$0 \to \mathbb{Z}_p \to \mathbb{T}_p \to \mathbb{R}/\mathbb{Z} \to 0$$

is a short exact sequence of topological groups.

It can be proved that for each $m \in \mathbb{Z} \geq 0$ such that $gcd(m, p) = 1$, the p-adic solenoid $\mathbb{T}_p$ has a unique cyclic subgroup of order m, and on the other hand that there is no element in $\mathbb{T}_p$ whose order is a power of p, namely, $\mathbb{T}_p$ has no p-torsion.

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