Cantor set

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1 background definition

DEFINITION 1.1. A metric space is a set X together with $d : X \times X \to \mathbb{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

DEFINITION 1.2 Let X be a metric space. X is a complete metric space, if $\forall$ $x \in X$ x is a limit points.
DEFINITION 1.3 Let X be a metric space. X is compact, if $\forall \{x_n\} \subseteq X, \exists x_{n_k} \to x_0$.
DEFINITION 1.4 Let X be a metric space. X is bounded, if $\forall x, y \in X, \exists M > 0$ let $d(x, y) < M$.

2 Cantor set

DEFINITION 2.1 Divide a interval $S_0 = [\,0, 1]$ into thirds, remove the open interval $\left[\frac{1}{3}, \frac{2}{3}\right]$which is the one in the middle, so the rest is $S_1 = \left[\,0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Then, do the same thing, divide each interval into thirds and remove the one in the middle. The set formed by the remaining points in the interval is called the Cantor set.

$$K = \bigcap_{n=1}^{\infty} S_n$$

Properties 1 *Z2 with the 2-adic norm is homeomorphic to the middle thirds Cantor set C with the norm it inherits from R.*

Proof 1 *Denote the Cantor set K in ternary. First, for $\frac{1}{3}$ in ternary, there's $\frac{1}{3} = \frac{x_1}{3^1} + \frac{x_2}{3^2} + \cdots$. Let's take $x_1 = 0$ and multiply both sides by 3^2, we have $3 = x_2 + \frac{x_3}{3^1} + \frac{x_4}{3^2} + \cdots$ x_2 can be $0, 1, 2$ when we plug in these three values, we get $x_2 = 2$. Then subtract 2 from both sides and multiply by 3, we can get $x_3 = 2$*

in a similar way. Then $\frac{1}{3}$ *in ternary equals to* $.222\cdots$
Now, denote $\forall x \in K$ *in ternary,* $x = 0.x_1x_2x_3\cdots$ *where* $x_i = 0$ *or* 2*. Let* S_i *be a closed interval.* $S_1 = [0, 0.22\cdots] \cup [2, 0.22\cdots] = I_{11} \cup I_{12}$
When $x \in I_{11}, x = 0$*;* $x \in I_{12}, x = 2$*.*
It can be summarized as $\forall x = 0.x_1x_2x_3\cdots \in K$ *if and only if* $x_i \in \{0, 2\} i = 1, 2, 3\cdots$*,On the other hand, any element* $z \in Z_2$ *has a* $2-adic$ *expansion.*

$$b = b_0 + b_1 2 + b_2 2^2 + \cdots + b_n 2^n$$

it's continuous with continuous inverse.

They Cantor set that tweaking the construction above are all total disconnected and compact, so that any point in the set is the limit of some nonstationary sequence contained in the set.

Properties 2 *Cantor sets are complete.*

Proof 2 $x = 0.x_1x_2x_3\cdots \in K$ *where* $x_i = 0or2$*. Define a sequence* $\{x_n\}, n = 1, 2, 3\cdots x_n \in [0, 1]$ *then* $|x_n - x| \leq \frac{2}{3^i}\sum_{i=n+1}^{\infty} = \frac{1}{3^n}$*,* $\lim_{n\to\infty}\frac{1}{3^n} = 0$*, so when* $n \to \infty, x_n \to x$*, which means that* x *is the limit point of* K *hence* K *is complete.*

Properties 3 *Cantor set is a bounded closed set.*

Proof 3 $\forall x = 0.x_1x_2x_3\cdots, y = 0.y_1y_2y_3\cdots \in K$ $x_i = y_i = 0or2$*.* (K, d) *is a metric space, from the construction of* K *there's* $M = 1$ *such that* $d(x_i, y_i) < M = 1$ *which means that* K *is bounded.*
Since the removed part $G = (\frac{1}{3}, \frac{2}{3},) \cup (\frac{1}{3^2}, \frac{8}{3^2}) \cup (\frac{7}{3^2}, \frac{8}{3^2}) \cup (\frac{1}{3^3}, \frac{2}{3^3}) \cup (\frac{7}{3^3}, \frac{8}{3^3}) \cup \cdots$ *is an open set, K=[0,1]* $\setminus G$ *is a closed set.*

Properties 4 *Cantor sets are uncountable sets.*

Proof 4 *Define* $h : [0, 1] \to K, \forall x \in [0, 1]$ *denote* $x = 0.x_1x_2\cdots$ *in binary form where* $x = 0or1$*. Then define* $h(x) = 0.y_1y_2\cdots$ *where* $y_i = 2x_i i = 1, 2, 3\cdots$*. Since* h *mapping to* K *and* $[0, 1]$ *is uncountable,* K *is uncountable.*

We can also construct a dence open set which the total length is $\alpha(0 < \alpha < 1)$. Take $p = \frac{1+2\alpha}{\alpha}$, construct it in a similar way to cantor Set.
First, remove the concentric open intervals of length $\frac{1}{p}$, then remove the concentric open intervals of length $\frac{1}{p^2}$ for both two parts that remain. Then do others in the same way, we'll get a open interval which is a open set T. The total length of T are

$$\sum_{n=1}^{\infty} 2^{n-1}(\frac{1}{p})^n = \frac{1}{p-2} = \alpha$$

$K_p = /T$ is a similar set to the Cantor set.