The Bruhat-Tits Tree

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Abstract

This paper discusses the construction of the Bruhat-Tits Tree. This is a tree whose vertices classify lattices in $\mathbb{Q}_p^2$, we compare it with lattices in $\mathbb{R}^2$ which are in bijection with the upper half-plane modulo the group $\mathrm{PSL}\,2(\mathbb{Z})$.

Contents

1 Introduction

The Bruhat–Tits tree's vertices classify lattices in $\mathbb{Q}_p^2$. It is an interesting analog of lattices in $\mathbb{R}^2$, which are in bijection with the upper half-plane modulo the group $\mathrm{PSL}_2(\mathbb{Z})$.

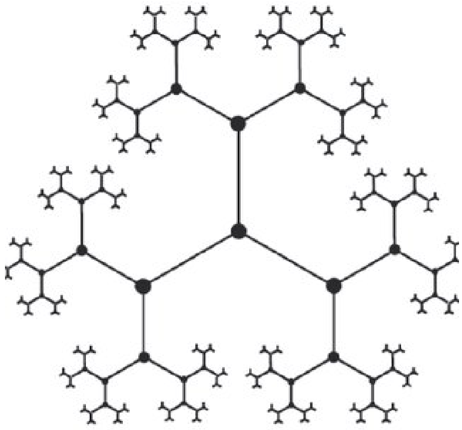

Figure 1. Bruhat-Tits Tree in $\mathbb{Q}_2$

One of it's more intriguing implications include a mathematical proof of the AdS/CFT correspondence. Using the tensor network living on the Bruhat-Tits tree one can give a concrete realization of the recently proposed p-adic AdS/CFT correspondence (a holographic duality based on the p-adic number field $\mathbb{Q}_p$). Instead of assuming the p-adic AdS/CFT correspondence, it can be shown how important features of AdS/CFT such as the bulk operator reconstruction and the holographic computation of boundary correlators are automatically implemented in this tensor network. This is a good (although somewhat long) resource to go on the same through after completing this paper. Url: `https://arxiv.org/pdf/1703.05445.pdf`.

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2 Preliminary Information

Definition 2.1 (General Linear Group)**.** A map f from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called *linear* if it maps a linear combination of vectors to the same linear combination of the images; that is

$$(u, v \in \mathbb{R}^m)(\lambda, \mu \in \mathbb{R}) \Rightarrow f(\lambda u + \mu v) \in \mathbb{R}^n \Rightarrow \lambda f(u) + uf(v) \in \mathbb{R}^n.$$

by fixing a basis $\{b_1, b_2, \ldots, b_n\}$ of the vector space $\mathbb{R}^n$, we can describe the effect of such a map by its matrix. $M_f = (a_{ij})$ where f maps the *ith* basis element b_i to $a_{i1}b_1 + a_{i2}b_2 + \ldots + a_{in}b_n$. Such a transformation is a bijection if it has an inverse map f^{-1} or equivalently if the determinant of its matrix is non-zero.

The set of all such invertible linear transformations from $\mathbb{R}^n$ to itself is called the ***General linear group***. Denoted by $\mathrm{GL}_n(R)$.

Definition 2.2 (Special Linear Group)**.** The set of all invertible transformations (or equivalently of invertible matrices) with determinant 1 is then a subgroup of $\mathrm{GL}_n(R)$ called the ***Special Linear group*** and denoted by $\mathrm{SL}_n(R)$.

Definition 2.3 (Projective Spaces)**.** We need to set up a geometry on a space different from $\mathbb{R}^n$. Taking the correspondence between points on two lines from a point P's projection not on either line. This is not bijective. A line from P parallel to the line m does not meet m and so no point on m corresponds to the point x on l. Similarly no point on l corresponds to the point y on m. To fix this we add a point at infinity to each line. Then $\lim\limits_{x\to\infty} m$ and $\lim\limits_{y\to\infty} l$. The space we get in this way consisting of the ordinary affine line $\mathbb{R}$ and this extra point is called the Real projective line and is written $\mathbb{R}P^1$.

Similarly we can set up a correspondence between the points of two planes π_1 and π_2. This time we add a line at infinity to each plane to make the correspondence into a bijection. This gives us a space consisting of the ordinary affine plane $\mathbb{R}^2$ together with this extra (projective) line at infinity which is called the Real projective plane and is written $\mathbb{R}P^2$.

Definition 2.4 (Projective Group)**.** If F is any field, the quotient group $\mathrm{GL}_n(F)\backslash\{\lambda I|\lambda \in F\backslash\{0\}\}$ is called the ***projective group*** and is written $\mathrm{PGL}_n(F)$.

Definition 2.5 (Projective Linear Group)**.** It is the induced action of the general linear group of a vector space V on the associated projective space $P(V)$; it is the quotient group: $\mathrm{PGL}(V) = \mathrm{GL}(V)/Z(V)$. Where $\mathrm{GL}(V)$ is the general linear group of V and $Z(V)$ is the subgroup of all nonzero scalar transformations of V; these are quotiented out because they act trivially on the projective space and they form the kernel of the action, and the notation "Z" reflects that the scalar transformations form the center of the general linear group.

It is also referred to as the Projective General Linear Group.

Definition 2.6. A partially ordered set in which every pair of elements has a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet) is called a ***lattice***.

An example is given by the power set of a set, partially ordered by inclusion, for which the supremum is the union and the infimum is the intersection.

Definition 2.7 (Non-Archimedean Metrics)**.** A metric induced by a *non-Archimedean norm* is a ***non-Archimedean metric***.

A non-Archimedean norm: $||x + y|| \leq \max(||x||, ||y||)(\forall x, y)$.

A non-Archimedean metric

(general): $d(x, y) \leq \max(d(x, z), d(z, y))(\forall x, y, z)$.

(for the mentioned norm): $d(x, y) = ||x - y|| = ||(x - z) + (z - y)|| \leq \max(||x - z||, ||z - y||) = \max(d(x, z), d(z, y))$.

Remark 2.8. Essentially, a non-Archimedean metric can be viewed as a metric induced by any absolute value that satisfies the ultrametric inequality.

Definition 2.9 (Algebraic Variety)**.** ***Algebraic Variety*** is the set of solutions of a system of polynomial equations over the real or complex numbers.

Definition 2.10 (Generic Fiber)**.** In algebraic geometry, a ***generic fiber*** or ***generic point*** P of an algebraic variety X is, roughly speaking, a point at which all generic properties are true, a generic property being a property which is true for almost every point.

This paper also makes use of several abbreviations, here is a table for your reference.

Symbol	**Usage**
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\lvert \cdot \rvert_p$	the p-adic absolute value.
$\mathbb{Q}_p$	the p-adic numbers
$\mathbb{P}$	projective spaces
$\mathbb{F}$	finite ields
iff	if and only if
ord	the order of a modulo n
red	reduction map
Spf	formal spectrum

3 Lattices and Planes

3.1 Lattices in $\mathbb{R}^2$

A lattice follows the following properties in $\mathbb{R}^n$:

1. It is a discrete additive subgroup of $\mathbb{R}^n$.
2. It is closed under addition and subtraction.
3. There exists a $\epsilon > 0$, such that any two distinct points in said lattice ($x \neq y$) at at least $||x - y|| \geq \epsilon$ away from each other.

The simplest lattice in $\mathbb{R}^n$ is the set of all n-dimensional vectors with integer entries, i.e, $\mathbb{Z}^n$. Integers can be added (and subtracted), and are at least $\epsilon = 1$ away from each other.

Other lattices can be easily obtained from $\mathbb{Z}^n$ by applying a (non-singular) linear transformation.

For instance, if $B \in \mathbb{R}^{k \times n}$ has full column rank[1], then $B(\mathbb{Z}^n) = \{Bx : x \in \mathbb{Z}^n\}$ is also a lattice.

[1]The columns of B are linearly independent

This set is closed under addition and subtraction. Additionally, if $B = [b_1, \ldots, b_k] \in \mathbb{R}^{n\times k}$ are linearly independent vectors in $\mathbb{R}^n$, then any point $y \in \text{span}(B)$ can be written as a unique linear combination $y = x_1b_1 + \ldots + x_nb_n$. Therefore, $y \in L(B)$ iff $\{x_1, \ldots, x_n\} \subseteq \mathbb{Z}$. Hence, the set B is also discrete.

Furthermore all lattices can be expressed as $B(\mathbb{Z}^n)$ for some B, so an equivalent definition of lattice is the following.

Definition 3.1. Let $B = [b_1, \ldots, b_k] \in \mathbb{R}^{n\times k}$ be linearly independent vectors in $\mathbb{R}^n$. The lattice generated by B is the set

$$\mathcal{L}(B) = \{Bx : x \in \mathbb{Z}^k\} = \left\{\sum_{i=1}^{k} x_i \cdot b_i : x_i \in \mathbb{Z}\right\}$$

of all the integer linear combinations of the columns of B. The matrix B is called a basis for the lattice $\mathcal{L}(B)$. The integers n and k are called the dimension and rank of the lattice respectively. And if $n = k$ then $\mathcal{L}(B)$ is called a full rank lattice.

3.2 The Upper Half-Plane

Definition 3.2. The ***Upper Half-Plane*** $\{H\}$ is the set of points (x, y) in the Cartesian plane with $y > 0$. It is also known as the ***Poincaré half-plane model***, and $\mathcal{H} = \{\langle x, y\rangle | y > 0; x, y \in \mathbb{R}\}$.

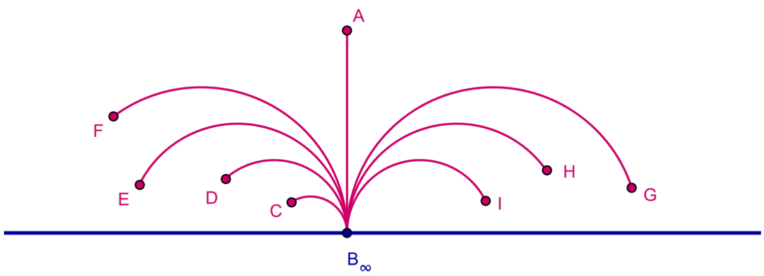

Figure 2. The Poincaré half plane model.

The metric of the model on the the half plane, $\{\langle x, y\rangle | y > 0\}$ is :

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

where s measures the length along a (possibly curved) line.

Remark 3.3. ds measures the distance 'travelled' on the x-axis for a very small (almost negligible) change in y. We are essentially taking the derivative of the distance, on the other hand, if one wanted to calculate the entire distance they would integrate this.

On this plane, $\text{PSL}_2(\mathbb{Z})$ acts bijectively maps the projectively extended rational line (the rationals with infinity) to itself, the irrationals to the irrationals, the transcendental numbers to the transcendental numbers, i.e. the upper half-plane to the upper half-plane.

3.3 Trees

Definition 3.4. In discrete mathematics, a ***graph*** refers to a structure amounting to a set of objects in which some pairs of the objects are in some sense "related". These objects correspond to mathematical abstractions called vertices, nodes or points. Each of the related pairs of vertices is called an edge.

There are several types of graphs; two major ones are cycle graphs and acyclic graphs.

Definition 3.5. Cycle graphs are connected graphs of the order $n \geq 3$, whose vertices can be denoted as $v_1, v_2, \ldots, v_n$, such that the edges are $\{v_i, v_{i+1}\}$ (where $i = \{1, 2, \ldots, n-1\}$) and $\{v_n, v_1\}$.

Where as ***acyclic graphs*** are graphs which have no cycles. A tree is a type of acyclic graphs.

Definition 3.6. A ***tree*** can be defined as a connected acyclic undirected graph. That is to say, a tree is a non-empty finite set of elements called vertices or nodes having the property that each node can have minimum degree 1 and maximum degree n.

Trees are often described as "visualization alternative to large cluttered concept lattices, which preserves all lattice entities and some of its structure." We will futher build on this idea with respect to the Bruhat-Tits Tree in the next sections.

3.4 Lattices in $\mathbb{Q}_p^2$

Definition 3.7. We call a subset $\mathcal{L} \subseteq \mathbb{Q}_p^2$ a lattice if $\mathcal{L}$ is a rank 2 free $\mathbb{Z}_p$-module of $\mathbb{Q}_p^2$.
Equivalently $\mathcal{L}$ is a lattice of $\mathbb{Q}_p^2$ if there exist 2 independent vectors $v_1, v_2 \in \mathbb{Q}_p^2$ such that

$$\mathcal{L} = \mathbb{Z}_p v_1 + \mathbb{Z}_p v_2$$

$$\mathcal{L} = \{x v_1 + y v_2 | x, y \in \mathbb{Z}_p\}.$$

Example. Lattices in $\mathbb{Q}_p^2$ are :
$\mathcal{L}_0 = \mathbb{Z}_2 = \mathbb{Z}_p(1,0) + \mathbb{Z}(0,1)$ and $\mathcal{L} = \mathbb{Z}_p(p^a, 0) + \mathbb{Z}(0, p^b), a, b \in \mathbb{Z}$

4 The Bruhat-Tits Tree

The Bruhat-Tits Tree is the combinatoric p-adic analog of the real half plane equipped with the natural action by $\mathrm{PSL}_2(\mathbb{Z})$. Its vertices correspond to $\{\mathbb{Z}_p - lattices \supseteq V_0 = \mathbb{Q}_p^2\}$ modulo $\mathbb{Q}_p^{\times}$ scaling. Two lattices $\mathcal{L}_1, \mathcal{L}_2$ form an edge iff after scaling $\mathcal{L}_1, \mathcal{L}_2$, the relation $p\mathcal{L}_1 \nsubseteq \mathcal{L}_2 \nsubseteq \mathcal{L}_1$ holds.
We formally define a Bruhat-Tits Tree as follows,

Definition 4.1. The Bruhat-Tits tree is the graph T , with vertices $[\mathcal{L}]$, where $[\mathcal{L}]$ is the equivalent class of some lattice $\mathcal{L}$ of $\mathbb{Q}_p^2$. There is an edge between two vertices v_1 and v_2 of T if and only if

$$(\exists \mathcal{L} s.t. v_1 = [\mathcal{L}]$$

$$(\exists \mathcal{L}' s.t. v_2 = [\mathcal{L}']$$

and

$$\mathcal{L} \supseteq \mathcal{L}' \subset p\mathcal{L}$$

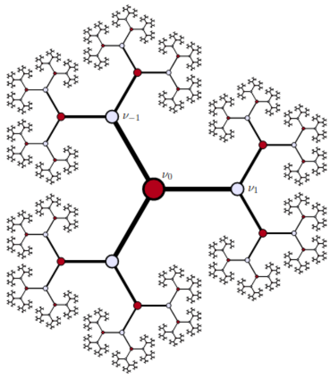

Figure 3. The Bruhat-Tits tree of SL(2)

We define the equivalence relation on the set of lattices of $\mathbb{Q}_p$ such that
$\mathcal{L} \sim \mathcal{L}' \iff \mathcal{L}' = \lambda\mathcal{L}$ for some $\lambda \in \mathbb{Q}_p^{\mathrm{x}}$.

Remark 4.2. $\mathcal{L} \supseteq \mathcal{L}' \subset p\mathcal{L}$ implies $\mathcal{L}' \supseteq \mathcal{L} \subset p\mathcal{L}'$, hence T is a undirected graph.

4.1 Constructing the Bruhat-Tits Tree in $\mathbb{Q}_p$

Fix a 2-dimensionaal $\mathbb{Q}_p$-vector space $V_0 \cong \mathbb{Q}_p^2$, and $\Omega = \mathbb{P}(V_0) - \mathbb{P}(V_0)(\mathbb{Q}_p) \cong \mathbb{P}^1_{\mathbb{Q}p} - \mathbb{P}^1(\mathbb{Q}p)$ with the action of $\mathrm{GL}(V_0) \cong \mathrm{GL}_2(\mathbb{Q}_p)$. For any lattice $\mathcal{L}$ in V_0, the generic fiber of $\mathbb{P}(\mathcal{L})$ is identified with $\mathbb{P}(V_0)$.

Choose an basis (e_1, e_2) of $\mathcal{L}$, then we can identify $\mathbb{P}(\mathcal{L})$ with $\mathbb{P}^1_{\mathbb{Z}p}$, and $\mathbb{P}(V_0)(\mathbb{C}_p) \overset{L}{\cong} \mathbb{P}^1(O_{\mathbb{C}_p})$.

Consider, $\hat{\Omega}_{\mathcal{L}}$ such that $(\mathbb{P}(\mathcal{L}) - \mathbb{P}^1(\mathbb{F}_p))^\vee$ over $\mathrm{Spf}\,\mathbb{Z} - p$, where $(-)^\vee$ means completion along the ideal (p). Its rigid generic fiber $\Omega_{\mathcal{L}}$ over $\mathbb{Q}_p$ is an open rigid sub-variety of Ω, with $\mathbb{C}_p$-points.

$$\Omega\mathcal{L}(\mathbb{C}p) = \hat{\Omega}\mathcal{L}(O_{\mathbb{C}p}) = \lim_n \hat{\Omega}\mathcal{L}(O_{\mathbb{C}p} \setminus \{p^n\}) = \mathbb{P}^1(\mathbb{C}p) - \mathrm{red}^{-1}(\mathbb{P}^1(\mathbb{F}_p)) \subseteq \Omega(\mathbb{C}p).$$

Where, red is the reduction map $\mathbb{P}^1(\mathbb{C}p)\mathcal{L} \cong \mathbb{P}^1(O_{\mathbb{C}p)} \to \mathbb{P}^1(\bar{\mathbb{F}}_p)$.

So ΩL is a complement of finitely many open discs in $\mathbb{P}(V_0)$.

More concretely, the basis (e_1, e_2) provides a pair of coordinates $[X_1, X_2]$ on $\mathbb{P}(V_0)$ i.e two sections of $O(1)$ that generates the line bundle $O(1)$. Let $T = X_1 \setminus X_2$ be the rational function on $\mathbb{P}(V_0)$ and its restriction on $D(X_2) = \mathbb{P}(V_0) - [0, 1]$ to be z, then we have

$$\hat{\Omega}_{\mathcal{L}} \cong Spf\mathbb{Z}_p[T, (T^p - T)^{-1}]^\vee$$

$$\Omega_{\mathcal{L}}(\mathbb{C}_p) = \{z \in \mathbb{C}_p : |z| = 1\} - \{z \in \mathbb{C}_p : |z - a| < 1, \text{ for some } a \in \mathbb{Z}_p\}.$$

It is then easy to see that for $z \in \Omega_L(\mathbb{C}p), |az + b| = \max\{|a|, |b|\}$ for any $a, b \in \mathbb{Q}_p$.

So $\Omega_{\mathcal{L}}(\mathbb{C}p)$ is $\mathrm{GL}(\mathcal{L}) \overset{e_1,e_2}{\cong} \mathrm{GL}_2(\mathbb{Z}_p)$-invariant.

Remark 4.3. We know by general construction that $g\Omega_{\mathcal{L}} = \Omega_{g\mathcal{L}}$ for any $g \in \mathrm{GL}(V_0)$. Notice that $\mathrm{GL}(V_0)$ acts transitively on all lattices, so we fix one lattice $M_0 = \mathbb{Z}_p e_{01} \bigoplus \mathbb{Z}_p e_{02} \subseteq V_0$ as the standard lattice, and let $[M_0 M_1]$ be the standard edge where $M1 = \mathbb{Z}_p e_{01} \bigoplus \mathbb{Z}_p p e_{02}$.

Proposition 4.4. *If $[L_1] \neq [L_2] \in BT_0$, then Ω_{L_1} doesn't intersect with Ω_{L_2}. Moreover,*

$$\bigcup_{[L]\in BT_0} \Omega_L(\mathbb{C}p) = \bigcup_{g\in \mathrm{GL}_2(\mathbb{Q}_p)} g\Omega_{M_0}(\mathbb{C}_p)$$

doesn't cover $\Omega(\mathbb{C}_p)$.

Proof. $\Omega(\mathbb{C}p)$ can be identified with the collection of $C^{\times}$-homothety classes of injective $\mathbb{Q}_p$-linear maps of V_0 into $\mathbb{C}_p$.

$z \in \Omega_L(\mathbb{C}_p)$ corresponds to the map $f : V_0 \to \mathbb{C}_p$ such that $f(e_1) = z$, $f(e_2) = 1$. As $|az+b| = \max\{|a|, |b|\}$ for any $a, b \in \mathbb{Q}_p$, we see $f^{-1}(O_{\mathbb{C}_p}) = \mathcal{L}$. This shows two different $\Omega_{\mathcal{L}}$ don't intersect. We have:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$$

and $z \in \Omega_{M_0}(\mathbb{C}_p), gz = \dfrac{az+b}{cz+d}, |gz| = \dfrac{|az+b|}{|cz+d|} = \dfrac{\max\{|a|+|b|\}}{\max\{|c|+|d|\}} \in \mathbb{Q}$. So any $z \in \mathbb{C}_p$ with $|z| \neq \mathbb{Q}$ is not in $\bigcup_{[L]\in BT_0} \Omega_{\mathcal{L}}(\mathbb{C}_p)$.

What's missing in the generic fiber can be seen below. Under the standard basis e_{01}, e_{02},

$$\Omega_{M_0}(\mathbb{C}_p) = \{z \in \mathbb{C}_p : |z| = 1\} - \bigcup_{a\in\mathbb{Z}_p} \{z \in \mathbb{C}_p : |z-a| < 1\},$$

$$\Omega_{M_1}(\mathbb{C}_p) = \{z \in \mathbb{C}_p : |z| = p^{-1}\} - \bigcup_{a\in p\mathbb{Z}_p} \{z \in \mathbb{C}_p : |z-a| < p^{-1}\}.$$

So if we define

$$\Omega_{M_0M_1}(\mathbb{C}_p) = \{z \in \mathbb{C}_p : p^{-1} \leq |z| \leq 1\} - \bigcup_{a\in p\mathbb{Z}_p - p\mathbb{Z}_p} \{z \in \mathbb{C}_p : |z-a| < 1\} - \{z \in \mathbb{C}_p : |z-a| < p^{-1}\}.$$

It will contain $\Omega_{M_0}(\mathbb{C}_p)$ and $\Omega_{M_1}(\mathbb{C}_p)$ and fill the "gap" between them naturally.

We must now formally define $\Omega_{M_0M_1}$.

Definition 4.5. For any edge $[L_1L_2] \in BT_1$, let $\mathbb{P}_{\mathcal{L}_1\mathcal{L}_2}$ be the blow up of $\mathbb{P}(\mathcal{L}_1)$ at the $\mathbb{F}_p$-point $pt_{\mathcal{L}_1\mathcal{L}_2} = \mathcal{L}_2/p\mathcal{L}_1 \in \mathbb{P}_{\mathcal{L}_1}(\mathbb{F}_p)$, which is equal to the blow up of $\mathbb{P}(\mathcal{L}_2)$ at the $\mathbb{F}_p$-point $p\mathcal{L}_1/p\mathcal{L}_2$. Its generic fiber is identified with $\mathbb{P}(V)$, and it special fiber has an unique singular point (still denoted by $pt\mathcal{L}_1\mathcal{L}_1$.

We define $\hat{\Omega}_{\mathcal{L}_1\mathcal{L}_2}$ as the formal scheme $(\mathbb{P}_{\mathcal{L}_1\mathcal{L}_2} - (\mathbb{P}_{\mathcal{L}_1\mathcal{L}_2}(\mathbb{F}_p) - pt_{\mathcal{L}_1\mathcal{L}_2}))^{\vee}$, and its rigid generic fiber over $\mathbb{Q}_p$ by $\Omega_{\mathcal{L}_1\mathcal{L}_2}$

Proposition 4.6.

$$\hat{\Omega}_{M_0M_1} = \mathrm{Spf}(\mathbb{Z}_p[T_0, T_1, (T_0^{p-1}-1)^{-1}, (T_1^{p-1}-1)^{-1}]/(T_0T_1 - p))^{\vee}$$

where $T_0 = X_2/X_1, T_1 = pX_1/X_2$. The embedding $\hat{\Omega}_{M_0} \hookrightarrow \hat{\Omega}_{M_0M_1}$ sends T_0 to T^{-1}, and T_1 to pT_0. In particular T_1 vanishes on $\hat{\Omega}_{M_{0,s}}$. The embedding $\hat{\Omega}_{M_1} \hookrightarrow \hat{\Omega}_{M_0M_1}$ sends T_0 to pT^{-1}, and T_1 to T. T_0 vanishes on $\hat{\Omega}_{M_{1,s}}$.

It's not hard to show $\Omega_{M_0M_1}(\mathbb{C}_p)$ agrees with the first definition by hand, and all $\Omega_{\mathcal{L}_1\mathcal{L}_2}$ cover Ω. Just as $\{z \in \mathbb{C}_\infty : |z| > 1, |Re(z)| \leq 1/2\}$ gives a fundamental domain of $\mathbb{H}_\infty$ for the action of $\mathrm{GL}_2(Z)$, one can think $\Omega_{M_0M_1}$ as a fundamental domain of the p-adic half plane for $\mathrm{GL}_2(\mathbb{Z}_p)xdiag\{\mathbb{Q}_p^{\times}, 1\}$.

Remark 4.7. For latter consideration of intersection theory, it's better to base change and assume the residue field is algebraically closed. We denote $W = W(\bar{\mathbb{F}_p})$ to be the ring of Witt vectors of $\bar{\mathbb{F}_p}$.

Theorem 4.8. *There is a natural regular* 2*-dimensional formal model* $\hat{\Omega}$ *over* $\operatorname{Spf}\mathbb{Z}_p$ *of* Ω*, inheriting the action of* $\mathrm{GL}(V_0) \cong \mathrm{GL}_2(\mathbb{Q}_p)$*. Moreover,*

1. *Its special fiber* Ω_s *is reduced, and is a union of projective lines* $\mathbb{P}_L$ *indexed by vertices of* BT*.* $\mathbb{P}_{L1}$ *and* $\mathbb{P}_{L2}$ *intersects iff* L_1, L_2 *form an edge, in which case they intersect transversally at the* $\mathbb{F}_p$*-point* $pt_{L_1L_2} = L_2/pL_1 \in \mathbb{P}_{L_1}(\mathbb{F}_p)$*.*

2. $\Omega \to \operatorname{Spf}\mathbb{Z}_p$ *is of strictly semi-stable reduction. In particular, for any point* $x \in |\Omega_W| = |\Omega_{W,s}|$*, if* x *in the intersection of two PL, the completed local ring* O_x *is isomorphic to* $W[[T_0, T_1]] \setminus (T_0T_1 - p)$*; if* x *is in* $\mathbb{P}_L - \mathbb{P}_L(\mathbb{F}_p)$ *for some* $\mathcal{L}$*, then* $O_x \cong W[[T]]$*;* x *is called a superspecial point and a ordinary point respectively.*

3. *If* $x = ptL_1L_2$ *is a superspecial point, then* $\hat{\Omega}_{L_1L_2}$ *is an affinoid open neighborhood of* x*; If* $x \in \mathbb{P}_L$ *is an ordinary point, then* $\hat{\Omega}_L$ *is an affinoid open neighborhood of* x*.*

4. *The action of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *is compatible with the action of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *on* BT*. In particular,* $g\mathbb{P}_L = \mathbb{P}_{gL}$*.*

Proof. Glue $\hat{\Omega}_{L_1L_2}$ along $\hat{\Omega}_L$, and note the Bruhat-Tits tree is connected so they glue together to $\hat{\Omega}$.

5 Conclusions and Moving Futher

Hence, we can see how the Bruhat-Tits Tree is the p-adic analog of lattices in $\mathbb{R}^2$, which are in bijection with the upper half-plane modulo the group $\mathrm{PSL}_2(\mathbb{Z})$. As interesting as this was, the Tree in itself has several baffling implications in p-adic string theory (the AdS/CFT correspondence to be specific).

One could also go on to explore the theory of building, which has important applications in several rather disparate fields. It is related to the the structure of reductive algebraic groups over general and local fields, and is used to study their representations. The results of Tits on determination of a group by its building have deep connections with rigidity theorems of George Mostow and Grigory Margulis, and with Margulis arithmeticity.

Special types of buildings are studied in discrete mathematics, and the idea of a geometric approach to characterizing simple groups proved very fruitful in the classification of finite simple groups. The theory of buildings of type more general than spherical or affine is still relatively undeveloped, but these generalized buildings have already found applications to construction of Kac–Moody groups in algebra, and to nonpositively curved manifolds and hyperbolic groups in topology and geometric group theory.

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