ON THE p-ADIC SOLENOID, ITS CONSTRUCTIONS AND VISUALIZATIONS

NICHOLAS JAMES, JINFEI HUANG, GURASEES SINGH DHANOA, KRISHNA PRANEETH SIDDE

Abstract. We analyze the properties of the p adic solenoid $\mathbb{S}_p$, a mathematical object that connects $\mathbb{R}$ and $\mathbb{Q}_p$ as a topological group. To do this, we look at $\mathbb{S}_p$ through the lenses of algebra and algebraic topology.

Contents

1. Introduction

In p-adic analysis, one learns about the structure of $\mathbb{Z}_p$ as an inverse limit, and building additional structures off of it such as $\mathbb{Q}_p$ and $\mathbb{C}_p$. However, not much is known about $\mathbb{S}_p$ even though it can be constructed in a similar way.

It is not an unknown structure, however. Indeed, the work of Alain Robert [Rob13] in his book *A Course in p-adic Analysis* provides one of the largest expositions on the p-adic solenoid that is currently known. On the other hand, there are gaps in the understanding of $\mathbb{S}_p$; Hofmann and Morris' book [HM06] *The Structure of Compact Groups* provides an inside look into the inner workings of Pontryagin Duality Theory and its ties to the character group of the solenoid.

What makes the p-adic solenod an important thing to note is its intimate relationship with the circle group; indeed, we will see later in this paper of how $\mathbb{S}_p$ can be expressed as an inverse limit of circle groups, similarly to how $\mathbb{Z}_p$ can be described as an inverse limit of the integers modulo p^n. Using circle groups in our construction allows for some convenient results with compactness and other theory developed in algebraic topology.

Ultimately, though, this approach is rather restrictive. It is more fulfilling from a visual perspective to see how $\mathbb{S}_p$ looks in three dimensions. Indeed, one can theoretically embed $\mathbb{S}_p$ into $\mathbb{R}^3$ in a certain why through wrappings around multiple tori. We propose one such construction for the curious.

Additionally, we also present the unorthodox construction of $\mathbb{S}_p$ as a quotient. While relatively unimportant in the general scheme of things p-adically, it is an alternative view of the solenoid that certainly should not be overlooked.

An important question is where exactly $\mathbb{S}_p$ fits in the p-adic mix. After all, we are familiar with objects such as $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ and even Ω_p, but how smaller or larger is $\mathbb{S}_p$ relative to any of those? We answer this question through the discussions of the subgroups of $\mathbb{S}_p$. In particular, we show that $\mathbb{S}_p$ contains both $\mathbb{Q}_p$ and $\mathbb{Z}_p$.

As one does for $\mathbb{Z}_p$ and $\mathbb{Q}_p$, we do not leave the topology of $\mathbb{S}_p$ unscathed. We show that it is both connected and compact, which follows rather easily from the formalism that we develop with inverse limits. Through this, some additional results follow immediately from point set topology, such as sequential and limit point compactness.

To discuss Pontryagin Duality Theory, one must develop some more theory. Due to the assumption of the reader's ability, we develop some basic theory on abelian groups, topological groups, and character groups. This is not all in vain, however; we use these prerequisites to connect all of the theory presented together into one idea: showing that $\mathbb{S}_p$ and its character group $\mathbb{Z}[1/p]$ exhibit Pontryagin Duality. This is not a special property of the solenoid, however. The abelian group $\mathbb{Z}_p$ and its character group $\mathbb{Z}(p^\infty)$ also exhibit Pontryagin Duality.

This work is inspired by both Hofmann and Morris' book, as well as the works of Robert. This expository primarily relies on their results. Our central aim is to provide a thorough exposition of $\mathbb{S}_p$, and combine all known knowledge as cohesively as possible. The authors of this work hope that this is as exhaustive as it can be.

2. Inverse Systems

The structure of $\mathbb{S}_p$ is based off of a very familiar mathematical object: the inverse system. Here, one defines the idea of a "limit" in an algebraic sense. While this is something usually reserved for analysis, we propose how to rigorously define this here based off of the exposition of Robert [Rob13].

2.1. Definition of Inverse System.

In order to fully understand the construction of $\mathbb{S}_p$ in an algebraic setting, we need to define the notion of an inverse (also known as "projective") system.

Definition 2.1. A sequence $(X_n, \varphi_n)_{n\geq 0}$ of sets X_n and transition maps $\varphi_n : X_{n+1} \to X_n$ is called an *inverse system*. A set X with maps $\psi_n : X \to X_n$ such that $\psi_n = \varphi_n \circ \psi_{n+1}$ for $n \geq 0$ is called the *inverse limit* (X, ψ_n) of the inverse system $(X_n, \varphi_n)_{n\geq 0}$ if the following condition holds: for each set A and mappings $f_n : A \to X_n$ satisfying $f_n = \varphi_n \circ f_{n+1}$, there is some unique factorization f of f_n such that for $n \geq 0$,

$$f_n = \psi_n \circ f : A \to X \to X_n.$$

This inverse limit (X, ψ_n) is denoted as $X = \varprojlim X_n$. (See Figure 1). When the mappings in the inverse system are surjective we say that it is *strict* and that the inverse limit is the *strict inverse limit*.

Remark 2.2. Definition 2.1 can be broken down significantly for clarification. Notice that there is a iterative behavior with f_n, or that

$$\begin{aligned} f_n &= \varphi_n \circ f_{n+1} \\ &= \varphi_n \circ (\varphi_{n+1} \circ f_{n+2}) \\ &\ \ \vdots \\ &= (\varphi_n \circ \varphi_{n+1} \circ \ldots \circ \varphi_{n+k}) \circ f_{n+k+1} \end{aligned}$$

which is just equivalent to $\psi_n \circ f$. So, by definition f is seen as a limit of each f_i as $i \to \infty$ and ψ_n is seen as a limit of the composition of the transition mappings φ_n. This can all be seen in the following diagram.

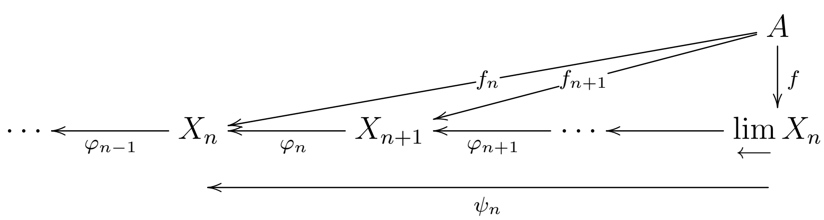

Figure 1. An inverse system $(X_n, \varphi_n)_{n\geq 0}$ together with its inverse limit (X, φ_n).

In addition, the "factorization" condition in Definition 2.1 is what is called a *universal property*; that is, inverse limits do not depend on the initial terms in the sequence of mappings; also notice that the universal property is responsible for making the diagram in Figure 1 commute. But to remove the ambiguity of this abstraction, we resort to an example.

2.2. **A Familiar Example.** While usually discussed in a p-adic analysis course, one common example of an inverse system equipped with an inverse limit should be very familiar. The inverse system $(\mathbb{Z}/p^n\mathbb{Z}, \varphi_n)_{n\geq 0}$ with transition maps (homomorphisms) $\varphi_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ forms the inverse limit $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, called the *$p$-adic integers*. (See Figure 2).

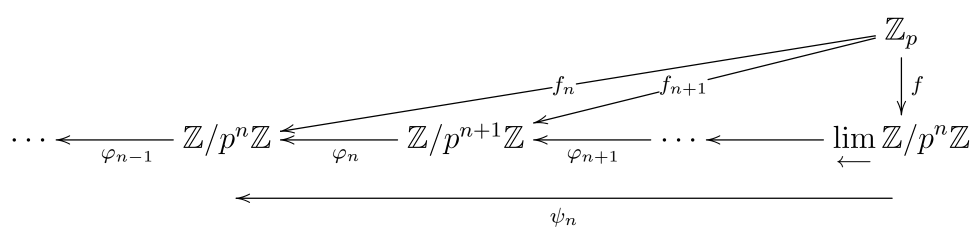

Figure 2. Diagram of $(\mathbb{Z}/p^n\mathbb{Z}, \varphi_n)_{n\geq 0}$. Visual depiction of $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

While this definition makes sense abstractly, one should informally think of $\mathbb{Z}_p$ as having an almost "fractal-like" behavior, with each iteration being a new congruence class modulo p^n. Indeed, this behavior is similar in nature to the inverse system, of which can be seen directly from definition.

2.3. **A Question of Existence.** A question that is typically asked in analysis is whether a limit exists in the formal sense. Given our higher levels of abstraction in the inverse system, one should certainly have this worry. The following theorem is meant to qualm any worries the reader may have pertaining to this issue.

Theorem 2.3. *For every inverse system $(X_n, \varphi_n)_{n\geq 0}$ there exists an inverse limit*

$$X = \varprojlim X_n \subset \prod_{n\geq 0} X_n$$

which is unique up to bijection.

Proof. We will first prove existence. Instead of defining X the way we previously did, let

$$X = \{(x_n)_{n\geq 0} \ : \ x_n \in X_n \text{ and } \varphi_n(x_{n+1}) = x_n \text{ for } n \geq 0\}$$

which obviously forms a subset of the product of each X_i. Let $\epsilon_n : X \to X_n$. If $x \in X$, then $\varphi_n(\epsilon_{n+1}(x)) = \epsilon_n(x)$ so if we restrict ϵ_n so that we have ψ_n map to X we have $\varphi_n \circ \psi_{n+1} = \psi_n$ as usual. Hence, the maps ψ_n and the set X can be viewed as an upper bound of the sequence (X_n) with respect to the transition maps φ_n. To finish the proof of existence, it remains to verify the universal property. Consider the mapping $\psi'_n : X' \to X_n$, where $\varphi_n \circ \psi'_{n+1} = \psi'_n$. We claim that there is a unique factorization of ψ'_n by ψ_n. Observe that

$$(\psi)'_n : X' \to \prod_{n\geq 0} X_n$$

and for each $u \in X'$ the mapping $u \mapsto (\psi'_n(u))$ holds. Because $\varphi_n(\psi'_{n+1}(u)) = \psi'_{n+1}(u)$ the map (ψ'_n) is contained in a subset of X. Hence there is a map

$$f : X' \to X \subset \prod_{n\geq 0} X_n$$

such that $\psi'_n = \psi_n \circ f$. This proves existence; now we prove uniqueness. Suppose that (X, ψ_n) and (X', ψ'_n) both exist and satisfy the universal factorization property. Then there is a map $f : X \to X'$ with $\psi_n = \psi'_n \circ f'$ and $\psi'_n = \psi_n \circ f$. Substituting the second into the first gives

$$\psi_n = (\psi'_n \circ f') \circ f = \psi'_n \circ f' \circ f = \psi'_n$$

so uniqueness holds as well. This completes our work. ∎

2.4. **The Topology of the Inverse System.** More depth in this subject occurs when one considers the possibility of $(X_n, \varphi_n)_{n\geq 0}$ being a topological space with continuous maps φ_n. By our previous construction, this means that $(X, \psi_n) = X = \varprojlim X_n$ forms a topological space as well, with mappings $\psi_n : X \to X_n$ that are also continuous. We prove the following important result.

Theorem 2.4. *An inverse limit of compact nonempty spaces is also nonempty and compact.*

Proof. Let $(K_n, \varphi_n)_{n\geq 0}$ be an inverse system composed of compact spaces. By Tychonoff's theorem, the product

$$\prod_{n\geq 0} K_n = K_0 \times K_1 \times \dots$$

is compact. Since the inverse limit K is contained in the above product, it follows that K is compact, as any closed subspace of a compact space is compact. Now we prove that K is nonempty. It suffices to show that all φ_n's are surjective. Define the chain of spaces

$$K_n' = \varphi_n(K_{n+1}) \supset K_n'' = \varphi_n(\varphi_{n+1}(K_{n+2})) = \varphi_n(K_{n+1}') \supset \dots$$

which are all compact and nonempty. Their intersection

$$\mathscr{L}_n = \bigcap_{k\geq 0} K_n^{(k)}$$

is nonempty in K_n. We see that $\varphi_n(\mathscr{L}_{n+1}) = \mathscr{L}_n$ and the restriction of φ_n to $\mathscr{L}_n$ leads to an inverse system with surjective transition mappings, completing the proof. ■

It follows from this theorem that the inverse limit of nonempty (finite) sets is nonempty. The condition that these sets be compact is just additional for larger results. But while on the topic of topological spaces, the subject of bases arises rather quickly.

Theorem 2.5. *Let X be an inverse limit of topological spaces X_n. The basis of the topology is furnished by the sets $\psi_n^{-1}(\mathcal{O}_n)$ for $n \geq 0$ and arbitrary open sets $\mathcal{O}_n \subset X_n$.*

Proof. Similar to before, define $x = (x_i)$ to be a sequence within the inverse limit X where

$$X = \{(x_n) \ : \ \varphi_n(x_{n+1}) = x_n \text{ for } n \geq 0\}.$$

We show that each $\mathcal{O}_n$ with $x \in \mathcal{O}_n$ forms a basis of neighborhoods. Let $\mathcal{O}_n \subset X_n$ and $\mathcal{O}_{n-1} \subset X_{n-1}$. Since it is assumed that $x_n \in \mathcal{O}_n$ and $x_{n-1} \in \mathcal{O}_{n-1}$ we have $\psi_n(x) = x_n$ where $x_n \in \mathcal{O}_n \cap \varphi_{n-1}^{-1}(\mathcal{O}_{n-1})$. This implies that $x \in \psi_n^{-1}(\mathcal{O}_n)$. Using the principle of mathematical induction, we see that we can show that an open set in

$$\prod_{N\geq n} \mathcal{O}_n \times \prod_{n>N} X_n$$

has an intersection with the inverse limit of the form $\psi_N^{-1}(\mathcal{O}_N)$ for some open set $\mathcal{O}_N \subset X_N$. ■

Our final result with inverse systems has to deal with topological closure, specifically the closure of a subset of the inverse limit.

Theorem 2.6. *Let $S \subset X = \varprojlim X_n$ of topological spaces X_n. The closure $\overline{S}$ of S is*

$$\overline{S} = \bigcap_{n\geq 0} \psi_n^{-1}(\overline{\psi_n(S)}).$$

Proof. First, we see that $S \subset \bigcap_{n\geq 0} \psi_n^{-1}(\overline{\psi_n(S)})$, which is closed, so $\overline{S}$ is within this intersection as well. On the other hand, let b be in this intersection. We show that it is also in $\overline{S}$. Let U be some neighborhood of b. From our previous results, we see that we can assume that $U = \psi_n^{-1}(\mathcal{O}_n)$ without loss of generality for some $\mathcal{O}_n \subset X_n$. This means that $\psi_n(b) \subset \mathcal{O}_n$ trivially. Since b was assumed to be in this intersection, we set $b \in \psi_n^{-1}(\overline{\psi_n(S)})$. Taking mappings gives $\psi_n(b) \in (\overline{\psi_n(S)})$. Hence, there is some $s \in S$ such that $\psi_n(s) \in \mathcal{O}_n$.

This means that we have $s \in S \cap \psi_n^{-1}(\mathcal{O}_n)$, showing that the neighborhood of b must meet S at some point, or that $\psi_n(b) \in S$. Done. ■

This implies an interesting corollary that we will mention. It has to deal with when a subset of an inverse limit is dense.

Corollary 2.7. *If S is a subset of $X = \varprojlim X_n$, then S is dense when all $\psi_n(S)$ are dense.*

Despite the relatively heavy amounts of formalism needed to define inverse systems, our work will soon yield fruit as the reader will soon see as they read on.

3. Topological Groups and Algebra

While we assume that the reader is familiar with the topics of groups, rings, and fields in the normal sense, deriving topological properties on these objects can prove rather difficult without defining new objects. This is what we seek to do in this section, in addition to deriving their properties. However, it begins rather unorthodox. To begin, we present a special group called the circle group, which will prove vital throughout the rest of this exposition.

3.1. The Circle Group.

We propose the following definition of an important additive quotient of groups.

Definition 3.1. The *circle group* $\mathbb{T}$ is defined as

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

However, we sometimes interchange the notation of $\mathbb{T}$ to express another algebraic relationship going on.

Theorem 3.2. *The circle group $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$.*

Proof. Take the function $\varphi : \mathbb{R} \to \mathbb{T}$ via the mapping $\theta \mapsto e^{i\theta} = \cos(\theta) + i\sin(\theta)$. To be clear, this is indeed an additive group homomorphism because

$$\begin{aligned}\varphi(\theta_1 + \theta_2) &= e^{i(\theta_1+\theta_2)} \\ &= e^{i\theta_1}e^{i\theta_2} \\ &= \varphi(\theta_1)\varphi(\theta_2).\end{aligned}$$

The kernel of this homomorphism is $2\pi\mathbb{Z}$, and by the First Isomorphism Theorem, we see that (upon rescaling a little),

$$\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{R}/\mathbb{Z} \cong \varphi(\mathbb{R}) = \mathbb{T}.$$

This completes the proof of the theorem. ■

Throughout the rest of this text, will will use $\mathbb{T}$ and $\mathbb{R}/\mathbb{Z}$ interchangebly. We hope that the reader sees the benefit of not committing to one notation.

3.2. **Topological Groups.** While we are already familiar with a group G, we can actually endow this group with a topology to turn it into something rather special.

Definition 3.3. A *topological group* G is a group endowed with a topology such that the mappings

$$\varphi : G \times G \to G, (x, y) \mapsto xy$$
$$\psi : G \to G, x \mapsto x^{-1}$$

are continuous mappings in G. We say that G is a *compact topological group* when its topology is compact Hausdorff, and a *locally compact topological group* when its topology is Hausdorff and the identity has a compact neighborhood. Lastly, we say that G is *discrete* if its topology is the discrete topology.

From the definitions above, it follows that both compact and discrete topological groups are also locally compact topological groups. It is relatively trivial to show that a subgroup of a topological group is also a topological group itself with respect to the subspace topology.

Example. The circle group $\mathbb{T}$ is a compact topological group with its usual topology. Any group G is a topological group with respect to the discrete topology.

It is rather fitting to introduce the mappings of one topological group to another. Not surprisingly, they behave similarly to mappings among groups in the normal sense.

Definition 3.4. A *morphism* of topological groups is a continuous mapping $\varphi : G \to H$ that is also a group homomorphism. It is said to be an *isomorphism* of topological groups if it has an inverse morphism of topological groups. We denote this isomorphism as $G \cong H$ per usual.

Of course, when studying topological groups we must define actions amongst them.

Definition 3.5. We say that a topological group G acts on a set X if there is a continuous function $\psi : G \times X \to X$ by the mapping $(g, x) \mapsto gx$ which implements a group action.

Using this language we can construct the quotient of two topological groups.

Definition 3.6. If $H \leq G$ then the set G/H of cosets $gH = \{gh \mid h \in H\}$ for each $g \in G$ is a topological space relative to the quotient topology called the quotient space of G modulo H. If $H \trianglelefteq G$ then G/H with respect to the quotient topology is called the *topological quotient group* of G modulo H.

It should not be of any concern to the reader of whether or not there are a finite amount of compact topological groups. The reason why is because of the following result.

Theorem 3.7. *If $\{G_j\}_{j \in J}$ is an indexed family of compact groups G_j, then the product*

$$G = \prod_{j \in J} G_j$$

with respect to the product topology is also a compact group. Additionally, any closed $H \leq G$ is also compact.

Proof. This follows directly from Tychonoff's Theorem. Indeed, $H \leq G$ implies the second claim directly given that G is compact. ∎

We conclude this subsection by observing when isomorphisms between topological groups occur.

Theorem 3.8. *If G is a topological group and H is a Hausdorff topological group, and if $f : G \to H$ is an injective morphism of topological groups, then the constriction $f' : G \to f(G)$ with $f'(g) = f(g)$ is an isomorphism.*

Proof. Since G is compact and H is Hausdorff, the image $f(G)$ is a compact group and f maps closed (hence compact) subsets of G onto compact (hence closed) subsets of H. Then f' being a bijective and closed map is a homeomorphism, and thus f^{-1} is a morphism of compact groups. ■

3.3. **Abelian Groups.** A large portion of studying the proceeding material is the study of abelian groups. Here are a couple of key facts that will be used later on.

Definition 3.9. If a group A is abelian, we define its *torsion* as

$$\mathrm{tor}(A) = \{a \in A \mid na = 0 \text{ for all } n \in \mathbb{N}\}$$

and we say A is *torsion-free* if $\mathrm{tor}(A) = \{0\}$.

Essentially, an abelian group is torsion-free if all of its elements have infinite order besides the trivial element. Again, the study of torsion groups can become rather involved (especially with free groups), but we only seek to analyze both of these ideas on the surface level.

Definition 3.10. If A and B are abelian groups, then we define

$$\mathrm{Hom}(A, B) = \{f \mid f : A \to B \text{ is a homomorphism}\} \subseteq B^A$$

as the set of (continuous) homomorphisms f from A to B. We call $\mathrm{Hom}(A, B)$ the *Hom group* of the abelian groups A and B.

Similarly to how we observed with inverse systems, we introduce one last definition that will help us link different algebraic objects together via a series of mappings.

Definition 3.11. For groups G_i, the chain of homomorphisms

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} G_n$$

is called an *exact sequence* when $\mathrm{image}(f_i) = \ker(f_{i+1})$. When $G_0 = G_n = 0$ we say that this is a *short exact sequence*.

Notice that these sets need not be groups, they can be rings or even modules as well. The definition of an exact sequence is by no means restricted to groups.

4. Character Groups

Using the theory developed in the previous section, we now have the tools necessary to learn the basic theory of character groups.

Definition 4.1. Let A be a (discrete) abelian group. Then $\mathrm{Hom}(A, \mathbb{T}) \subseteq \mathbb{T}^A$ is called the *character group* of A which we denote as $\widehat{A}$. Its elements are called the *characters* of A.

It is important to note that $\widehat{A}$ is an abelian group with respect to pointwise addition. Seeing this definition in light of Theorem 3.7, we have the following result.

Theorem 4.2. *The character group of an abelian group is a compact and abelian.*

This allows us to have surprisingly weird relationships between groups.

Example. Consider the function $\varphi : \widehat{\mathbb{Z}} = \operatorname{Hom}(\mathbb{Z}, \mathbb{T}) \to \mathbb{T}$ by the mapping $f \mapsto f(1)$. This mapping is an isomorphism and continuous via pointwise convergence. Since $\widehat{\mathbb{Z}}$ is compact and $\mathbb{T}$ is Hausdorff, by Theorem 3.2 this is an isomorphism so we have $\widehat{\mathbb{Z}} \cong \mathbb{T}$.

It is certainly helpful to discover some properties of the elements of $\widehat{A}$. We propose the following definition to get started.

Definition 4.3. Let X and Y be sets and $F \subseteq Y^X$ the set of functions from X to Y. We say that F *separates* the points of X if for any $x_1, x_2 \in X$ that there is an $f \in F$ such that $f(x_1) \neq f(x_2)$.

We can use this idea of separation in sets to see that in an abelian group A, there are enough characters to separate the points. We propose another definition.

Definition 4.4. Let A be an abelian group. Then A is said to be *divisible* if for every $a_1 \in A$ and $n \in \mathbb{N}$ that there is an $a_2 \in A$ such that $n \cdot a_2 = a_1$.

Remark 4.5. This definition is essentially saying that for each element in the abelian group, that each element is a multiple of another. Simple examples of this would be $\mathbb{R}$ and $\mathbb{Q}$.

The important takewaway from this is that given some subgroup S of A, for some homomorphism $f : S \to I$ where I is divisible, then there is a homomorphic extension $F : A \to I$. It can be seen in this diagram.

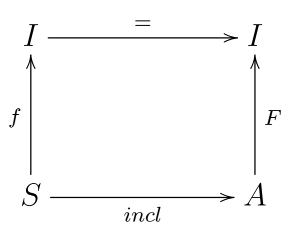

Figure 3. Diagram of homomorphic extension of divisibility maps.

Theorem 4.6. *The characters of an abelian group A separate the points.*

Proof. Let a be some nonzero element in A. We find some character $\chi \in \widehat{A} = \operatorname{Hom}(A, \mathbb{T})$ such that $\chi(a) \neq 0$. Let $S \subseteq \mathbb{Z}(a)$ in A. If S is infinite then S is free, and for any nonzero $t \in \mathbb{T}$ there is an $f : S \to \mathbb{T}$ such that $f(a) = t$. If S is finite, let $|S| = n$. Then $S \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{T}$, which means that $f : S \to \mathbb{T}$ is injective. Now, we can let $\chi : A \to \mathbb{T}$ be a homeomorphic extension with respect to the divisible group $\mathbb{T}$, or in other words that $\chi(a) = f(a) \neq 0$. Done. ∎

Using this as a sort of "stepping stone" to the rest of the good stuff about character groups, we propose the following.

Theorem 4.7. *Let A be an abelian group and let $\widehat{A} = \operatorname{Hom}(A, \mathbb{T})$ be its character group with characters χ. Then the function*

$$\eta_A : A \to \hat{\hat{A}},\ \eta_A(a)(\chi) = \chi(a)$$

is an injective morphism of groups and if G is a compact abelian group then

$$\eta_G : G \to \hat{\hat{G}},\ \eta_G(g)(\chi) = \chi(g)$$

is a morphism of compact abelian groups.

Proof. We prove the results of η_A first. Notice that the fact that it's a morphism follows immediately from pointwise-addition in $\widehat{A}$. An element $k \in \ker(\eta_A)$ if $\chi(k) = 0$ for all $\chi \in \widehat{A}$. However, since all such χ seperate the points in A, we must have that $k = 0$. This implies that $\ker(\eta_A)$ is trivial, which implies that η_A is injective.

Now we prove the results of η_G. From the same logic before it follows that η_G is a morphism. The issue is that we need continuity now, since we are dealing with compact groups. The character map $\chi : G \to \mathbb{T}$ by $g \mapsto \chi(g)$ is continuous by the continuity of characters. Hence, we have

$$(\chi)_{\chi \in \widehat{G}} : G \to \mathbb{T}^{\widehat{G}},\ g \mapsto (\chi)_{\chi \in \widehat{G}}(g)$$

is also continuous via the product topology. Since, by definition, we have that the character group of the character group $\hat{\hat{G}} = \mathrm{Hom}(\widehat{G}, \mathbb{T}) \subseteq \mathbb{T}^{\widehat{G}}$ inherits its structure from the product, we have that η_G is continuous. ■

5. The p-adic Solenoid $\mathbb{S}_p$

After all of that build up, it is time to put it all together. We begin with our discussion of $\mathbb{S}_p$ where we started: inverse systems.

Definition 5.1. The *p-adic solenoid* $\mathbb{S}_p$ is the inverse limit of the inverse system $(\mathbb{R}/p^n\mathbb{Z}, \varphi_n)_{n \geq 0}$. In other words,

$$\mathbb{S}_p = \varprojlim \mathbb{R}/p^n\mathbb{Z}.$$

To clarify what is going on here, it is helpful to revisit the example of $\mathbb{Z}_p$. We viewed $\mathbb{Z}_p$ as the inverse limit of the integers modulo p^n; in the case of $\mathbb{S}_p$, we are looking at the inverse limit of circle groups $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ in a similar way. With this definition, we can actually

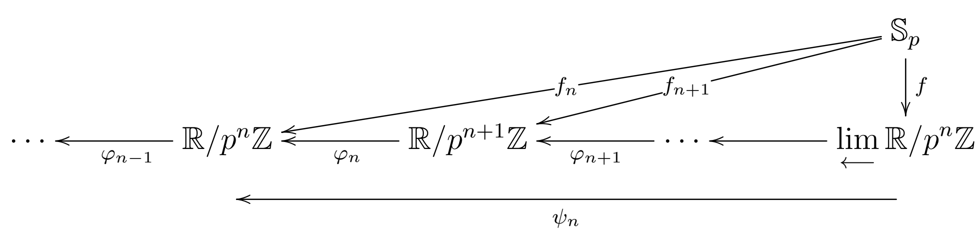

Figure 4. Diagram of $(\mathbb{R}/p^n\mathbb{Z}, \varphi_n)_{n \geq 0}$. Visual depiction of $\mathbb{S}_p = \varprojlim \mathbb{R}/p^n\mathbb{Z}$.

approach the solenoid in a visual way as an infinite embedding of tori in $\mathbb{R}^3$. Take a solid torus D_1 and embed it into Euclidean space. Then take a second solid torus, say D_2, and wind it around D_1 a p amount of times. Then do the same for D_3 to D_2, and D_{n+1} for D_n. As n grows larger and larger, we continue this recursion and form the intersection

$$D_\infty = \bigcap_{i=1}^{\infty} D_i$$

and we have that D_∞ is homeomorphic to $\mathbb{S}_p$. A visualization of this is shown below for the case of $p = 2$, which we call the *dyadic solenoid* $\mathbb{S}_2$.

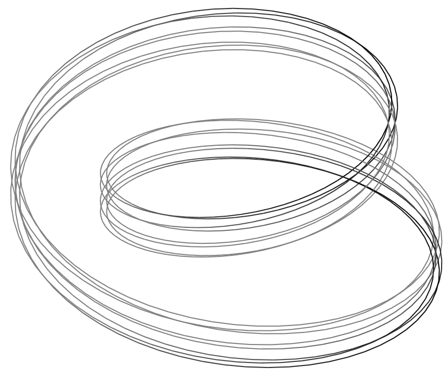

Figure 5. An embedding of $\mathbb{S}_2$ into $\mathbb{R}^3$.

This construction of the dyadic solenoid is helpful in visualizing for general p in a way that is more appealing to human intuition.

6. p-torsion of $\mathbb{S}_p$

Before we begin with the discussion of p-torsion for $\mathbb{S}_p$, we will look at the unique cyclic subgroups of $\mathbb{S}_p$ to prove that $\mathbb{S}_p$ has no p-torsion. This section is based off of the works of Robert [Rob13].

Theorem 6.1. *The solenoid $\mathbb{S}_p$ has a unique cyclic subgroup C_m of order m for each positive integer $m \geq 1$ prime to p.*

Proof. The proof of this theorem is followed from the well-known fact that there exists a unique cyclic subgroup of order m in the circle $m^{-1}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$ for every positive integer $m \geq 1$. Using this, let us denote temporarily by C_m^n the cyclic subgroup of order m of the circle $\mathbb{R}/p^n\mathbb{Z}$, whose subgroup is $m^{-1}\mathbb{Z}/p^n\mathbb{Z}$. Since the transition maps

$$\varphi_n : \mathbb{R}/p^{n+1}\mathbb{Z} \longrightarrow \mathbb{R}/p^n\mathbb{Z}$$

have a kernel of order p prime to m by assumption, they induce isomorphisms $C_m^{n+1} \to C_m^n$. The *inverse limit* of this constant sequence is therefore the cyclic subgroup $C_m \subset \mathbb{S}_p$.

To prove the uniqueness of this cyclic subgroup C_m, we consider some homomorphism $\sigma : \mathbb{Z}/m\mathbb{Z} \to \mathbb{S}_p$. Then the composite map

$$\psi_n \circ \sigma : \mathbb{Z}/m\mathbb{Z} \to \mathbb{S}_p \to \mathbb{R}/p^n\mathbb{Z}$$

has an image in the unique cyclic subgroup C_m^n of the circle $\mathbb{R}/p^n\mathbb{Z}$. Hence, σ has an image in C_m, and this concludes the proof. ■

This unique cyclic subgroup C_m of order m for each positive integer $m \geq 1$ prime to p has image $\psi(C_m)$ in the circle given by

$$\psi(C_m) = m^{-1}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}.$$

Since $\psi^{-1}(m^{-1}\mathbb{Z}/\mathbb{Z}) \cong C_m \times \mathbb{Z}_p$, C_m is the maximal finite subgroup contained in $\psi^{-1}(m^{-1}\mathbb{Z}/\mathbb{Z})$.

Using this information about the unique cyclic subgroup C_m of $\mathbb{S}_p$, we will show that $\mathbb{S}_p$ has no p-torsion. We extend the definition of p-torsion of some abelian group from 3.9 by noting that

$$\mathrm{tor}_p(A) = \{a \in A \mid p^n a = 0 \text{ for all } n \in \mathbb{N}\}$$

Theorem 6.2. *The p-adic solenoid $\mathbb{S}_p$ has no p-torsion.*

Proof. Let $\sigma : \mathbb{Z}/p\mathbb{Z} \to \mathbb{S}_p$ be any homomorphism of a cyclic group of order p into the solenoid. We claim that all composites

$$\varphi_n \circ \psi_{n+1} \circ \sigma : \mathbb{Z}/p\mathbb{Z} \to \mathbb{S}_p \to \mathbb{R}/p^{n+1}\mathbb{Z} \to \mathbb{R}/p^n\mathbb{Z}$$

are trivial. Indeed, the composite map

$$\psi_{n+1} \circ \sigma : \mathbb{Z}/p\mathbb{Z} \to \mathbb{S}_p \to \mathbb{R}/p^{n+1}\mathbb{Z}$$

must have an image in the unique cyclic subgroup of order p of the circle $\mathbb{R}/p^{n+1}\mathbb{Z}$, and this subgroup is precisely the kernel of the connecting homomorphism φ_n and $\psi_n \circ \sigma = \varphi_n(\psi_{n+1} \circ \sigma)$. Consequently, there is no element of order p in $\mathbb{S}_p$ and a fortiori no element of order p^k for $k \geq 1$ in $\mathbb{S}_p$. This concludes that $\mathbb{S}_p$ has no p-torsion. ■

7. The Quotient $\mathbb{S}_p$

In order to represent the p-adic solenoid as a quotient, we need a way of embedding $\mathbb{R}$ and $\mathbb{Q}_p$ into $\mathbb{S}_p$. The following two theorems state this more precisely, which are based off of the work of [Rob13].

Theorem 7.1. *The solenoid $\mathbb{S}_p$ contains a dense subgroup isomorphic to $\mathbb{R}$.*

Proof. The idea of the proof is rather simple: find an injective homomorphism from $\mathbb{R}$ to $\mathbb{S}_p$, and restrict to its image. Such a homomorphism is obtained from the universal property in Definition 2.1. Observe that the projections $f_n : \mathbb{R} \to \mathbb{R}/p^n\mathbb{Z}$ given by $x \mapsto x \pmod{p^n\mathbb{Z}}$ are such that $f_n = \varphi_n \circ f_{n+1}$, where φ_n is a transition map of $(\mathbb{R}/p^n\mathbb{Z}, \varphi_n)$. Hence there is a unique map $f : \mathbb{R} \to \mathbb{S}_p$ with the factorization $f_n = \psi_n \circ f : \mathbb{R} \to \mathbb{S}_p \to \mathbb{R}/p^n\mathbb{Z}$. Since $(\mathbb{R}/p^n\mathbb{Z}, \varphi_n)$ is a system of groups $\mathbb{R}/p^n\mathbb{Z}$ and homomorphisms φ_n, we see that f (and ψ_n) is also a homomorphism.

We will prove that the kernel of f is trivial. Suppose that x is a nonzero real number. Choose n large enough such that $p^n > |x| > 0$. This means that $f_n(x) \neq 0 + \mathbb{R}/p^n\mathbb{Z}$, which implies $f(x) \neq 0 \in \mathbb{S}_p$ since $f_n(x) = \psi(f(x))$; it follows that x is not in $\ker f$. Thus f is injective, as desired.

Since f_n is surjective and $f_n = \varphi_n \circ f$, we have

$$\psi_n(\mathrm{image}(f)) = \mathrm{image}(f_n) = \mathbb{R}/p^n\mathbb{Z}.$$

It follows by Corollary 2.7 that $\mathrm{image}(f) \subseteq \mathbb{S}_p$ is dense. ■

Theorem 7.2. *The solenoid $\mathbb{S}_p$ contains a dense subgroup isomorphic to $\mathbb{Q}_p$.*

Proof. Define the subgroups

$$H_k = \psi_0^{-1}(p^k\mathbb{Z}/\mathbb{Z}) \subset \mathbb{S}_p.$$

In particular, we see that $H_0 = \ker \psi_0 = \mathbb{S}_p$. This is a subgroup of index p^k of H_k, as

$$H_k = \varprojlim p^k\mathbb{Z}/p^n\mathbb{Z} \cong p^{-k}\mathbb{Z}_p.$$

Therefore,

$$\mathbb{Q}_p \cong \psi^{-1}(\mathbb{Z}[1/p]/\mathbb{Z}) = \bigcup \psi_0^{-1}(p^{-k}\mathbb{Z}/\mathbb{Z}) = \bigcup H_k \subset \mathbb{S}_p.$$

The density of $\bigcup H_k \subset \mathbb{S}_p$ follows from the density of the images

$$\psi_n\left(\bigcup H_k\right) = \mathbb{Z}[1/p]/p^n\mathbb{Z} \subset \mathbb{R}/p^n\mathbb{Z}$$

and Corollary 2.7. ■

Theorem 7.3. *The sum homomorphism* $f : \mathbb{R} \times \mathbb{Q}_p \longrightarrow \mathbb{S}_p$ *furnishes an isomorphism* $f' : (\mathbb{R} \times \mathbb{Q}_p)/\Gamma_p \cong \mathbb{S}_p$ *both algebraically and topologically.*

For the proof we will have to introduce fractional and integral parts of the p-adic integers.

Definition 7.4. For any p-adic integer represented as $x = \sum_{i \geq m} x_i p^i$ starting at index $m = v_p(x) \in \mathbb{Z}$, we define

$$[x] = \sum_{i \geq 0} x_i p^i \in \mathbb{Z}_p \text{ as the integral part of } x,$$

$$\langle x \rangle = \sum_{i < 0} x_i p^i \in \mathbb{Z}[1/p] \text{ as the fractional part of } x$$

We thus obtain

$$x = [x] + \langle x \rangle : \mathbb{Q} = \mathbb{Z}_p + \mathbb{Z}[1/p]$$

Also we see if $\langle x \rangle \neq 0$, then $\langle x \rangle = ap^v$ for integers a and $v < 0$. *decomposition depends on the choice of representation chosen for digits; here* $0 \leq x_i \leq p - 1$. With this choice, more can be said of the fractional part as a real number, namely

$$0 \leq \langle x \rangle = \sum_{i<0} x_i p^i = \sum_{1 \leq j \leq -v} \frac{x_{-j}}{p^j} < (p-1) \sum_{j \geq 1} \frac{1}{p^j} = 1.$$

Hence, the fractional part of any p-adic number satisfies the usual condition of

$$\langle x \rangle \in [0, 1) \cap \mathbb{Z}[1/p].$$

Now we can proceed with the proof of Theorem 7.3 using the knowledge of fractional and integral parts.

Proof. Since all the maps f_n are surjective (2.1), the map f has a dense image. Moreover, using integral and fractional parts introduced before,

$$f(t, x) = f(t + \langle x \rangle, x - \langle x \rangle) = f(s, y),$$

where $s \in \mathbb{R}$ and $y = x - \langle x \rangle = [x] \in \mathbb{Z}_p$. Again, we have,

$$f(s, y) = f(s - [s], y + [y]) = f(u, z),$$

where $u = s - [s] \in [0, 1)$ and $z = y + [s] \in \mathbb{Z}_p$. This proves,

$$\text{image}(f) = f(\mathbb{R} \times \mathbb{Q}_p) = f([0, 1) \times \mathbb{Z}_p).$$

A fortiori, the image of f is equal to $f([0, 1] \times \mathbb{Z}_p)$, hence is compact and closed. Consequently, f is surjective (and f' is bijective). In fact, the preceding equalities also show that the

Hausdorff quotient (recall that the subgroup Γ_p is discrete and closed) is also the image of the compact set $\Omega = [0,1]\times\mathbb{Z}_p$ and hence is compact. This results in the continuous bijection

$$f' : (\mathbb{R}\times\mathbb{Q}_p)/\Gamma_p \longrightarrow \mathbb{S}_p$$

which forms a mapping of two compact spaces, and thus f' is a homeomorphism. ■

Corollary 7.5. *$\mathbb{S}_p$ is a quotient of $\mathbb{R}\times\mathbb{Z}_p$ by the discrete subgroup $\Delta z = \{(m,-m); m\in\mathbb{Z}\}$*

$$f' : (\mathbb{R}\times\mathbb{Z}_p)/\ker f' \cong \mathbb{S}_p.$$

Proof. Since the restriction of the sum homomorphism $f : \mathbb{R}\times\mathbb{Q}_p \longrightarrow \mathbb{S}_p$ to the subgroup $\mathbb{R}\times\mathbb{Z}_p$ is already surjective, this restriction gives a (topological and algebraic) isomorphism

$$f' : (\mathbb{R}\times\mathbb{Z}_p)/\ker f' \cong \mathbb{Z}_p.$$

But, we see that

$$\ker f' = (\ker f)\cap(\mathbb{R}\times\mathbb{Z}_p) = \Delta z = \{(m,-m) : m\in\mathbb{Z}\}$$

and the proof is complete. ■

Presentations of the solenoid can be gathered in a commutative diagram of homomorphisms:

$$\begin{array}{ccc} \mathbb{Z}[1/p] & \longrightarrow & \mathbb{Q}_p \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & \mathbb{S}_p. \end{array}$$

Figure 6. Presentation of $\mathbb{S}_p$

Corollary 7.6. *$\mathbb{S}_p$ can also be viewed as a quotient of the topological space $[0,1]\times\mathbb{Z}_p$ by equivalence relation identifying $(1,x)$ to $(0,x+1)$ $(x\in\mathbb{Z}_p)$.*

Proof. This follows immediately from the previous corollary, since the restriction of sum homomorphism to $[0,1]\times\mathbb{Z}_p$ is already surjective, whereas its restriction to $[0,1)\times\mathbb{Z}_p$ is bijective. ■

This gives the same topological model as in Figure 5: *two ends of the cylinder $[0,1]\times\mathbb{Z}_p$ having basis $\mathbb{Z}_p$ by a twist representing the unit shift of $\mathbb{Z}_p$.*

8. Closed subgroups of $\mathbb{S}_p$

This section is based off of the work of [Rob13]. We first discuss a few theorem that help us in proving the existence of certain closed subgroups of $\mathbb{S}_p$. By the end of the section, we hope to establish all the closed subgroups of $\mathbb{S}_p$.

Theorem 8.1. *Let $\sigma : C_{p^m} \to C_{p^{m-1}}$ be a* surjective homomorphism *between two cyclic groups of order p^m and p^{m-1}. Then the only subgroup $H \subset C_{p^m}$ not contained in the* kernel *of σ is $H = C_{p^m}$.*

Proof. Recall that any subgroup of a cyclic group is cyclic and that the number of generators of $C_n \cong \mathbb{Z}/n\mathbb{Z}$ is given by the Euler φ-function $\varphi(n)$. If $n = p^m$, the power of a prime p, the number of generators is

$$\varphi(p^m) = p^{m-1}(p-1) = p^m - p^{m-1}.$$

Consequently, all elements not in the kernel of a *surjective homomorphism* of a cyclic group of order p^m onto a cyclic group of order p^{m-1} are generators of the cyclic group of order p^m which implies that the kernel has an order of p^{m-1}. ■

Theorem 8.2. *For each integer $k \geq 0$, there is exactly one subgroup $H_k \subset \mathbb{S}_p$ having a projection of order p^k in the circle: $\psi(H_k) = p^{-k}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$. This subgroup is $H_k = \psi^{-1}(p^{-k}\mathbb{Z}/\mathbb{Z}) \subset \mathbb{S}_p$.*

Proof. The proof of this theorem is quite simple as it is directly followed from the aforementioned theorem. If we the aforementioned theorem to each surjective homomorphism $p^{-k}\mathbb{Z}/p^{n+1}\mathbb{Z} \to p^{-k}\mathbb{Z}/p^n\mathbb{Z}$ in the sequence of connecting homomorphisms defining the solenoid as an inverse limit, the inverse limit of these cyclic groups is $p^{-k}\mathbb{Z}_p$. ■

We now look at the closed subgroups of $\mathbb{S}_p$ in the following theorem.

Theorem 8.3. *The closed subgroups of the solenoid $\mathbb{S}_p$ are*
1. *C_m, the cyclic group of order m relatively prime to p ($m \geq 1$),*
2. *$C_m \times p^k\mathbb{Z}_p$, where m is prime to p and $k \in \mathbb{Z}$,*
3. *$\mathbb{S}_p$ itself (connected).*

Proof. Let us consider H to be a closed subgroup of $\mathbb{S}_p$. From the fact that H is compact, it is implied that image $\psi(H)$ is a closed subgroup of the circle $\mathbb{R}/\mathbb{Z}$. But from the previous theorems, this stands true only when

$$\psi(H) = n^{-1}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} \text{ cyclic of order } n \geq 1$$

or

$$\psi(H) = \mathbb{R}/\mathbb{Z} \text{ is the whole circle.}$$

Through these possibilities, we prove the existence of various closed subgroups in $\mathbb{S}_p$.

1. From the second case where $\psi(H) = \mathbb{R}/\mathbb{Z}$ is the whole circle, $\psi_n(H) \subset \mathbb{R}/p^n\mathbb{Z}$ must be a closed subgroup of *finite index* and must be open in this circle. By *connectivity*, $\psi_n(H) \subset \mathbb{R}/p^n\mathbb{Z}$ and since this must hold for all $n \geq 1$, we conclude that
$$H = \bar{H} = \bigcap_{n\geq 1} f_n^{-1}((\bar{f_n}H)) = \mathbb{S}_p.$$
Evidently, $H = \mathbb{S}_p$ in this case.
2. Suppose $\psi(H) = \{0\}$, then $H \subset \psi^{-1}(0) = \mathbb{Z}_p \subset \mathbb{S}_p$. The only possibilities now are:
$$H = \{0\}, \quad p^k\mathbb{Z}_p \text{ for some integer } k \geq 0.$$
However, these possibilities already occur in the list for $C_m = \{0\}$ when $(m = 1)$.
3. Let us now consider the case when $\psi(H) = a^{-1}\mathbb{Z}/\mathbb{Z}$ is cyclic and nontrivial. Supposed $a = p^k \cdot m$ with $k \geq 0$ and m prime to p. Then, by the Chinese remainder theorem, this cyclic group $\psi(H)$ us a direct product of the cyclic subgroups $m^{-1}\mathbb{Z}/\mathbb{Z}$ and $p^{-k}\mathbb{Z}/\mathbb{Z}$. If $k \geq 1$, theorem 8.1 shows that $\psi_{n+1}(H)$ must contain an element of order p^{k+1}, and theorem 8.2 shows that H contains $\psi^{-1}(p^k\mathbb{Z}/\mathbb{Z}) = p^{-k}\mathbb{Z}_p \subset \mathbb{S}_p$. Finally, $H = C_m \times p^{-k}\mathbb{Z}_p$. However, when $k = 0$, two possibilities occur: Either

$\psi_n(H)$ is cyclic of order m for all n, or there is a first n such that $\psi_n(H)$ contains an element of order p. In the first case, we note that $H = C_m$ and in the second case, $H = C_m \times p^n\mathbb{Z}_p$.

This concludes the proof of this theorem. ■

9. Topology of $\mathbb{S}_p$

We have started to make progress towards an understanding of topology of the $\mathbb{S}_p$ since Theorem 2.4. We also have a basic visualisation of the $\mathbb{S}_p$ from Figure 5 and Corollary 7.6.

Lemma 9.1. *Let $\mathbb{U}$ be any proper subset of the circle $\mathbb{R}/\mathbb{Z}$. Then the subspace $\psi^{-1}(\mathbb{U}) \subset \mathbb{S}_p$ of the solenoid is homeomorphic to $\mathbb{U} \times \mathbb{Z}_p$. The map*

$$(t, x) = (t - [t], x + [t]) \longrightarrow (0, x + [t])$$

furnishes by restriction a continuous retraction of $\psi^{-1}([0, \eta]) \subset \mathbb{S}_p$ onto neutral fiber $\mathbb{Z}_p \subset \mathbb{S}_p (0 < \eta < 1)$.

Proof. We have a continuous surjective homomorphism $\psi : \mathbb{S}_p \longrightarrow \mathbb{R}/\mathbb{Z}_p$ leading to $\mathbb{S}_p$ being expressed in the chain

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{S}_p \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0.$$

The subspaces $\psi^{-1}([0, \eta])(0 < \eta < 1)$ have continuous retractions on the fiber of $\mathbb{Z}_p$ since $\psi^{-1}([0, \eta])$ is homomorphic to $[0, \eta] \times \mathbb{Z}_p$. ■

Lemma 9.2. *$\mathbb{S}_p$ is connected.*

Proof. We have to prove that the closed subgroups of $\mathbb{S}_p$ are $\mathbb{S}_p$ only. Assume that $\varphi(H) = a^{-1}\mathbb{Z}/\mathbb{Z}$ is cyclic and not trivial. We can write $a = p^k \cdot m$ with $k \leq 0$ and m relatively prime to p. We see that this cyclic group is the direct product of two subgroups $m^{-1}\mathbb{Z}/\mathbb{Z}$ and $p^{-k}\mathbb{Z}/\mathbb{Z}$. If $k \geq 1$, then $\varphi_{n+1}(H)$ must contain an element of order p^{k+1}. We also see that H contains $\varphi^{-1}(p^{-1}\mathbb{Z}/\mathbb{Z}) = p^{-k}\mathbb{Z}_p \subset \mathbb{S}_p$, and finally $H = C_m \times p^{-k}\mathbb{Z}_p$. If $k = 0$, two possibilities arise, either $\varphi_n(H)$ is cyclic of order m for all n, or there is a first n such taht this group $\varphi_n(H)$ contains an element of order p. In the first case $H = C_m$, while in the second case $H = C_m \times p^n\mathbb{Z}_p$. ■

Lemma 9.3. *$\mathbb{S}_p$ is a compact space.*

Proof. From Definition 5 we know that

$$\mathbb{S}_p = \varprojlim \mathbb{R}/p^n\mathbb{Z}$$

which is the inverse limit of the inverse system $(\mathbb{R}/p^n\mathbb{Z}, \varphi_n)_{n \geq 0}$. Applying Theorem 2.4 we see that $\mathbb{S}_p$ is compact. ■

We also note that $\mathbb{S}_p$ shows another interesting topological property. We introduce the following definition to help.

Definition 9.4. A compact and connected topological space K is called indecomposable when the only partition of K in two compact and connected subsets is the trivial one.

It turns out that the solenoid is indecomposable, which we prove here.

Lemma 9.5. *$\mathbb{S}_p$ is indecomposable.*

Proof. Let there be two compact connected subsets A and B assuming $A \neq \mathbb{S}_p$. We have that

$$K = \bigcap_{n \geq 1} \psi_n^{-1}(\psi_n(K))$$

for every compact set K, $\psi_n(A) \neq \mathbb{R}/p^n\mathbb{Z}_p$ for some integer. We take an n and $b \in B$ such that,

$$\varphi_n^{-1}(b) \subset \mathbb{R}/p^{n+1}\mathbb{Z}.$$

We see there is a restriction on φ_n as

$$C = \varphi_n^{-1}(\varphi_n(B)) = \varphi_{n+1}(B).$$

We have to prove $\varphi|_C$ not being injective implies $\varphi|_C$ is surjective under the assumptions. It is enough to do so when $C \neq \mathbb{R}/a\mathbb{Z}$. Take a point $P \notin C \subset \mathbb{R}/a\mathbb{Z}$ and consider a projection from the point P of the circle $\mathbb{R}/a\mathbb{Z}$ onto a line $\mathbb{R}$. We see this is a homeomorphism

$$f : (\mathbb{R}/a\mathbb{Z}) - \{P\} \longrightarrow \mathbb{R}$$

The image $f(C)$ of the subset C is a connected subset of the real line containing the two different images of two different congruent points mod $\mathbb{Z}$. Since any connected set in the real line is an interval, this proves that $f(C)$ contains the whole interval linking these two congruent points. This means that C contains a whole arc I of the circle with the image $\varphi(I) = \mathbb{R}/\mathbb{Z}$. This completes the proof. ■

Corollary 9.6. *The p-adic solenoid is an indecomposable compact connected topological space.*

10. The Pontryagin Duality of $\mathbb{S}_p$

This section is largely based on the works of Hofmann and Morris [HM06]. Our goal now is to find the Pontryagin Dual of $\mathbb{S}_p$. To do this, we combine the language of inverse systems and abelian groups into one cohesive theory, and propose a quick rephrasing of the definition of an inverse system.

Definition 10.1 (Topological Groups)**.** Let D be a directed set. An *inverse system* of topological groups G over D is a family of morphisms (bonding maps)

$$\mathcal{I} = \{f_{jk} : G_k \to G_j \mid (i,j) \in D \times D, j \leq k\}$$

where G_i, G_j are topological groups with $j, i \in D$ satisfying the following criteria:

(1) $f_{ij} = e_{G_j}$, for all $j \in D$.
(2) $f_{jk} \circ f_{kl} = f_{jl}$ for all $j, k, l \in D$ with $j \leq k \leq l$.

The inverse limit of $\mathcal{I}$ is denoted as $\varprojlim \mathcal{I}$.

We leave it to the reader to discern that both of the definitions used to describe the inverse system are equivalent. Indeed, it is also beneficial to see why these bonding maps are surjective when these groups are compact. As an authors note, we assume all abelian groups are discrete topological unless otherwise specified.

Let A be an abelian group and let Υ be its family of subgroups. This actually forms a directed set: if $E, F \in \Upsilon$ are arbitrary subgroups, then $F + E \in \Upsilon$ as well. Simply put, A is just the union of all $F \in \Upsilon$. However, if $E, F \in \Upsilon$ and $E \subseteq F$ then the inclusion map from E

to F induces a morphism of both of their character groups $f_{EF} : \widehat{F} \to \widehat{E}$ via $f_{EF}(\chi) = \chi|E$. In other words, f_{EF} takes some character $\chi \in \widehat{F}$ and maps it restrictively to a character in $\widehat{E}$. In total, the family of morphisms

$$\mathcal{I} = \{f_{EF} : \widehat{F} \to \widehat{E} \mid E, F \in \Upsilon, E \subseteq F\}$$

forms an inverse system of compact abelian groups. Using the divisibility of $\mathbb{T}$, each character on E extends to one on F which means that $\mathcal{I}$ is strict (or that all f_{EF} are surjective). The inclusion map from F to A induces an injective morphism with $f_{FA} : \widehat{A} \to \widehat{F}$ with $f(\chi) = \chi|F$.

Theorem 10.2. *The mapping* $\Theta : \widehat{A} \to \varprojlim \widehat{F}_{F\in\Upsilon}$ *defined by* $\chi \mapsto (\chi|F)_{F\in\Upsilon}$ *is an isomorphism of compact abelian groups.*

Proof. We first have to show that this is a morphism above everything else. We can see this by noting that this is the same as writing

$$\Theta : \mathrm{Hom}(A, \mathbb{T}) \to \varprojlim_{F\in\Upsilon} \mathrm{Hom}(F, \mathbb{T}), \chi \mapsto (\chi|F)_{F\in\Upsilon}$$

which is a morphism of compact groups by Theorem 4.2. By definition, an element $\chi \in \ker(\Theta)$ iff $(\chi|F)_{F\in\Upsilon} = 0$. This only happens when $\chi = 0$, so $\ker(\Theta)$ is trivial and Θ is injective. To prove surjectivity, let $\alpha = (\chi_F)_{F\in\Upsilon} \in \varprojlim \widehat{F}_{F\in\Upsilon}$. We show that $\Theta(\chi) = \alpha$. By the definition of the bonding maps that form the inverse system, for every pair of finitely generated subgroups $E \subseteq F$ in A we have $\chi_F|E = \chi_E$. Now we define $\chi : A \to \mathbb{T}$ as $\chi(a) = \chi_F(a)$ for some $a \in F \subseteq A$ (we can do this since $\chi_F(a) \in \mathbb{T}$ does not depend on F). If we take $F = \mathbb{Z}(a) + \mathbb{Z}(b)$ for $a, b \in A$, then

$$\chi(a + b) = \chi_F(a + b) = \chi_F(a) + \chi_F(b) = \chi(a) + \chi(b)$$

so $\chi \in \mathrm{Hom}(A, \mathbb{T})$ and $\Theta(\chi) = \chi|F = \chi_F$. This shows that $\Theta(\chi) = \alpha$, which means that Θ is surjective. Since we proved injectivity and surjectivity, we have that bijectivity follows so we are done. ■

Similarly to showing how the union of sets with some property obtains a set with that property, we prove a result with characters.

Theorem 10.3. *If G is a strict inverse limit of* $\varprojlim_{j\in D} \widehat{G_j}$ *then*

$$\widehat{G} = \bigcup_{j\in D} \widehat{G_j}.$$

Proof. We prove this using a containment argument. If we assume G is a strict inverse limit of comapact abelian groups $\varprojlim_{j\in D} \widehat{G_j}$ with bonding maps $f_j : G \to G_j$, then every character $\chi : G_j \to \mathbb{T}$ gives another character $\chi \circ f_j : G \to \mathbb{T}$. Since each f_j is surjective, $\chi \circ f_j : G \to \mathbb{T}$ with the map $\chi \mapsto \chi \circ f_j$ is injective. Thus, $\widehat{G_j} \subseteq \widehat{G}$ so the first direction is proved.

Now we prove the second part. Assume that $\chi : G \to \mathbb{T}$ is in $\widehat{G}$. If we denote the image of an open interval $(-1/n, 1/n)$ in $\mathbb{T}$ as V, then $\{0\}$ is the only subgroup of $\mathbb{T}$ that is contained in V as well. Now, because $\mathcal{O} = \chi^{-1}(V)$ is an open neighborhood around 0 in G, we have $\ker(f_j) \subseteq \mathcal{O}$ for some $j \in D$, which means that $\ker(f_j)$ is a subgroup of $\mathbb{T}$ contained in V

and thus is $\{0\}$. From this we get $\ker(f_j) \subseteq \ker(\chi)$ and that there is a unique morphism $\chi_j : G_j \to \mathbb{T}$ such that $\chi = \chi_f \circ f_j$. Hence, we have $\widehat{G_j} \supseteq \widehat{G}$ so the proof is complete. ■

Now it is time to prove the major theorem that we have working up to for quite a while now. We prove the first half of the Pontryagin Duality Theorem.

Theorem 10.4 (Pontryagin Duality Theorem)**.** *For any abelian group A, the morphism $\eta_A : A \to \hat{\hat{A}}$ is an isomorphism. Equivalently, $\eta_A : A \to \mathrm{Hom}(\widehat{A}, \mathbb{T})$ is an isomorphism.*

Proof. As we previously proved in Theorem 10.2, we know that $\widehat{A}$ is the strict inverse limit of $\lim_{F\in\Upsilon} \widehat{F}$ with Υ being a directed family of finitely generated subsets of A. The limit maps $f_F : \widehat{A} \to \widehat{F}$ are given by $f_F(\chi) = \chi|F$, which are surjective and induce injective morphisms (inclusion maps)

$$\mathrm{Hom}(f_F : \mathbb{T}) : \mathrm{Hom}(\widehat{F}, \mathbb{T}) \to \mathrm{Hom}(\widehat{A}, \mathbb{T}),$$

with $\mathrm{Hom}(f_F, \mathbb{T})(\Xi) = \Xi \circ f_F$. Because of Theorem 10, we already know that $\mathrm{Hom}(\widehat{A}, \mathbb{T})$ is the union of the images of $\mathrm{Hom}(f_F, \mathbb{T})$. Thus, for any $\Omega \in \mathrm{Hom}(\widehat{A}, \mathbb{T})$, there is an $F \in \Upsilon$ such that Ω is in the image of $\mathrm{Hom}(f_F, \mathbb{T})$. Additionally, there is a $\Xi \in \mathrm{Hom}(\widehat{F}, \mathbb{T})$ such that $\Omega = \mathrm{Hom}(f_F, \mathbb{T})(\Xi) = \Xi \circ f_F$. We can now apply a lemma.

Lemma 10.5. *If F is a finitely generated abelian group, then $\eta_F : F \to \mathrm{Hom}(\widehat{F}, \mathbb{T})$ is an isomorphism.*

Thus, there is an $a \in F$ such that $\Xi = \eta_F(a) \in \mathrm{Hom}(\widehat{F}, \mathbb{T})$. Hence, $\Omega = \eta_F(a) \circ f_F$. To finish proving surjectivity, we notice that for any $\chi : A \to \mathbb{T}$ that

$$\Omega(\chi) = \eta_F(a)(f_F(\chi)) = \eta_F(a)(\chi|F) = (\chi|F)(a) = \chi(a) = \eta_A(a)(\chi).$$

This proves that $\eta_A : A \to \hat{\hat{A}}$ is surjective. We already proved that η_A was injective in Theorem 4.7, so η_A is bijective and the proof is complete. ■

$$\begin{array}{ccc} F & \xrightarrow{\eta_F} & \mathrm{Hom}(\widehat{F}, \mathbb{T}) \\ {\scriptstyle \mathrm{incl}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(\widehat{\mathrm{incl}}, \mathbb{T})} \\ A & \xrightarrow{\eta_A} & \mathrm{Hom}(\widehat{A}, \mathbb{T}) \end{array}$$

Figure 7. Commutative diagram of the Pontryagin Duality Theorem

Note that when G is a compact abelian group and satisfies the above theorem, that G is said to have *duality*. However, the result that we just found is not the complete story. Indeed, there is one last theorem we must proof.

Theorem 10.6. *If G is a compact abelian group is $\lim_{j\in D} G_j$ of a strict inverse system and each G_j has duality then G has duality as well.*

Proof. We first show that $\eta_A : G \to \hat{\hat{G}}$ is bijective. Note that we still have to do this despite the Pontryagin Duality Theorem, since we are assuming that G is ablelian *and* compact. In light of Theorem 4.7, we actually have to prove that this is surjective and injective this time as well.

Assume that $\Omega \in \hat{\hat{G}}$, or that $\Omega \in \mathrm{Hom}(\widehat{G}, \mathbb{T})$. By Theorem 10, we know that we can write

$$\widehat{G} = \bigcup_{j \in D} \widehat{G_j}.$$

Denote $\Omega_j = \Omega | \widehat{G_j}$. Then $\Omega_j \in \mathrm{Hom}(\widehat{G}, \mathbb{T})$. By hypothesis we are assuming that G_j has duality, so η_{G_j} is surjective and thus there is a $g_j \in G_j$ such that $\eta_{G_j}(g_j) = \Omega_j$. We claim that $g = (g_j)_{j \in D} \in \prod_{j \in D} G_j$ is an element of G. Without loss of generality, let $k \leq j$ for $j, k \in D$. We have the following (commutative) diagram:

$$\begin{array}{ccc} G_k & \xrightarrow{\eta_{G_k}} & \mathrm{Hom}(\widehat{G_k}, \mathbb{T}) \\ {\scriptstyle f_{jk}} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(\widehat{f_{jk}}, \mathbb{T})} \\ G_j & \xrightarrow{\eta_{G_j}} & \mathrm{Hom}(\widehat{G_j}, \mathbb{T}) \end{array}$$

However, notice that $\mathrm{Hom}(\widehat{f_{jk}}, \mathbb{T})$ is a restriction map sending Ω_k to $\Omega_k | \widehat{G_j} = \Omega_j$. Thus we find that

$$\eta_{G_j}(f_{jk}(g_k)) = \hat{\hat{f}}_{jk}(\eta_{G_j}(g_k)) = \hat{\hat{f}}_{jk}(\Omega_k) = \Omega_j = \eta_{G_j}(g_j).$$

But since each G_j was assumed to have duality, we have that η_{G_j} is injective (because it is bijective), so we have $f_{jk}(g_k) = g_j$. This shows that $g \in \lim_{j \in D} G_j$.

Next, consider the limit map $f_j : G \to G_j$. We also have a diagram similar to before:

$$\begin{array}{ccc} G & \xrightarrow{\eta_{G}} & \mathrm{Hom}(\widehat{G}, \mathbb{T}) \\ {\scriptstyle f_{j}} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(\widehat{f_{j}}, \mathbb{T})} \\ G_j & \xrightarrow{\eta_{G_j}} & \mathrm{Hom}(\widehat{G_j}, \mathbb{T}) \end{array}$$

Hence, $\hat{\hat{f}}_j(n_G(g)) = \eta_{G_j}(f_j(g)) = \eta_{G_j}(g_j) = \Omega_j$. However, this time we see that $\mathrm{Hom}(\widehat{f_j}, \mathbb{T})$ is the restriction $\Xi \to \Xi | \widehat{G_j} = \Omega$. Hence, $\eta_G(g) = \Omega$ and surjectivity is proven.

To finish we show that η_G is injective. This is rather simpler than proving surjectivity but we still show it. Note that this is equivalent to saying that the characters G separate the points; consider the set $\mathcal{K} = \{\ker(f_j) \mid j \in D\}$. Clearly we have $\cap \mathcal{K} = \{0\}$ so there is a $j \in D$ such that $g \notin \ker(f_j)$. Since G_j was assumed to have duality by hypothesis, its characters separate the points. We are done at this point, since this means that there is some $\chi \in \widehat{G_j}$ such that $\chi(f_j(g)) \neq 0$, and thus $\chi \circ f_j \in \widehat{G}$ is a character of G that does not annihilate g. Our proof is now complete. ■

All of this work amounts to a very happy ending indeed. We present the following result as a conclusion to our work.

Theorem 10.7. *The p-adic solenoid $\mathbb{S}_p$ has duality and its character is $\mathbb{Z}[1/p]$.*

Proof. Given that $\mathbb{S}_p$ is the inverse limit of circle groups, one must simply show that $\widehat{\mathbb{T}} = \mathbb{Z}$ which is not hard. Thus, $\mathbb{S}_p$ has duality. To find its dual, we note that the dual to the morphism $\mu_p : \mathbb{T} \to \mathbb{T}$ is $\mu_p : \mathbb{Z} \to \mathbb{Z}$ which is equivalent to $\mathbb{Z} \to \frac{1}{p}\mathbb{Z}$. By Theorem 10.3 we see then that we must have

$$\widehat{\mathbb{S}_p} = \bigcup_{n \geq 1} \frac{1}{p^n}\mathbb{Z} = \mathbb{Z}[1/p].$$

This completes our work. ■

An elightening thing to note is how $\mathbb{Q}_p$ is intimately related with the solenoid. Define

$$\Phi_n : \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}} \to \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}} \ , \ \Phi_n(q + p^{n+1}\mathbb{Z}) = q + p^n\mathbb{Z}.$$

Then applying this we see that we can describe the p-adic field as

$$\mathbb{Q}_p = \lim \left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{\mathbb{Z}} \xleftarrow{\Phi_1} \frac{\frac{1}{p^\infty}\mathbb{Z}}{p\mathbb{Z}} \xleftarrow{\Phi_2} \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^2\mathbb{Z}} \xleftarrow{\Phi_3} \cdots \right).$$

Not only do we have $\mathbb{Z}[1/p] \subset \mathbb{Q}_p$, but we showed early that $\mathbb{Q}_p$ is a dense subset of $\mathbb{S}_p$. This provides an interesting "size issue" between the three structures.

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