p-ADIC INTERPOLATION OF THE RIEMANN ZETA FUNCTION

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Abstract.
The Riemann ζ function is a popular function, especially due to the famous open problem related to it, known as the Riemann Hypothesis. In this paper, we provide a way to represent this function p-adically.

1. Introduction

We first introduce the Riemann ζ-function.

Definition 1.1. The Riemann ζ function is the function $\zeta : \mathbb{C} \to \mathbb{C}$ such that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We might wonder how we can turn this function into a function over the p-adic numbers. To do this, we need to first understand the behavior of the function itself.

2. The ζ-Function for Negative Odd Integers

First, we consider the properties of the ζ function itself. We first find a closed formula for the ζ function evaluated at negative odd integers.

Proposition 2.1. *The ζ-function satisfies the functional equations*

$$\zeta(1-s) = \frac{2\cos(\pi s/2)\Gamma(s)}{(2\pi)^s}\zeta(s)$$

and

$$\zeta(2k) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!}\left(-\frac{B_{2k}}{2k}\right).$$

Proof of this fact is outside the scope of this paper and will thus be omitted; however, a discussion of this is present in [KR15].

Proposition 2.2. *We have*

$$\zeta(1-k) = -B_k/k$$

for nonnegative even integers k.

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Proof. Let $k = 2z$. Then

$$\begin{aligned}
\zeta(1-2z) &= \frac{2\cos(\pi z)\Gamma(2z)}{(2\pi)^{2z}}\zeta(2z) \\
&= \frac{2\cos(\pi z)\Gamma(2z)}{(2\pi)^{2z}}(-1)^z\pi^{2z}\frac{2^{2z-1}}{(2z-1)!}\left(-\frac{B_{2z}}{2z}\right) \\
&= \frac{2(-1)^z(2z-1)!}{(2\pi)^{2z}}(-1)^z\pi^{2z}\frac{2^{2z-1}}{(2z-1)!}\left(-\frac{B_{2z}}{2z}\right) \\
&= \left(-\frac{B_{2z}}{2z}\right) \\
&= -\frac{B_k}{k}
\end{aligned}$$

as desired. ■

With these properties, we can start to translate our function from $\mathbb{C}$ to $\mathbb{Z}_p$.

3. p-adic Interpolation of the ζ-function

Definition 3.1. We let ζ_p satisfy

$$\zeta_p(1-k) = (1-p^{k-1})\zeta(1-k) = (1-p^{k-1})\left(-\frac{B_k}{k}\right)$$

for $k = 2, 4, 6, \ldots$.

Our goal now is to translate the term on the right into something involving elements of $\mathbb{Z}_p$. To do this, we can use integration over $\mathbb{Z}_p$, which uses integration as developed in measure theory. However, it is developed slightly differently in this context, as we will show.

Definition 3.2. Let X be a compact open subset of $\mathbb{Q}_p$. A p-adic distribution on X is a map μ such that μ is additive and takes the set of compact-opens of X to $\mathbb{Q}_p$.

Similarly, we can define an integral as follows.

Definition 3.3. Let U be a compact open. We define the integral

$$\int \mathbb{1}_U\, \mu = \mu(U),$$

where $\mathbb{1}_U$ is the characteristic function of U.

Before we consider our p-adic ζ function further, we need to first investigate the Bernoulli numbers.

Definition 3.4. We let the *Bernoulli polynomials* satisfy

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}.$$

We let the map $\mu_{B,k}$ on intervals $a + p^N$ satisfy

$$\mu_{B,k}(a+p^N) = p^{N(k-1)}B_k\left(\frac{a}{p^N}\right).$$

Last, we let $\mu_{k,\alpha}$ be the *regularized Bernoulli distribution*, defined by

$$\mu_{k,\alpha}(U) = \mu_{B,k}(U) - \alpha^{-k}\mu_{B,k}(\alpha U).$$

We note that by basic properties of the integral we have

$$\int_{\mathbb{Z}_p} \mathbb{1}_{\mu_{B,k}} = \mu_{B,k}(\mathbb{Z}_p) = B_k.$$

We want to find some relationship between the measures $\mu_{B,k}$.

Proposition 3.5. *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ satisfy $f(x) = x^{k-1}$ for some fixed positive integer k. Let X be a compact-open subset of $\mathbb{Z}_p$. Then*

$$\int_X \mathbb{1}_{\mu_{k,\alpha}} = k \int_X f_{\mu_{1,\alpha}}.$$

Proof. We note that

$$\mu_{k,\alpha}(a + p^N) \equiv ka^{k-1}\mu_{1,\alpha}(a + p^N) \pmod{p^{N-\operatorname{ord}_p d_k}},$$

where d_k is the least common multiple of the coefficients of $B_k(x)$. Now assume N is large, so that X is a union of intervals of the form $a + p^N$. Then we have

$$\begin{aligned}\int_X \mathbb{1}_{\mu_{k,\alpha}} &= \sum_{0\le a<p^N : a+p^N\in X} \mu_{k,\alpha}(a + p^N) \\ &= \sum_{0\le a<p^N , a+p^N\in X} ka^{k-1}\mu_{1,\alpha}(a + p^N) \pmod{p^{N-\operatorname{ord}_p d_k}} \\ &= k \sum_{0\le a<p^N , a+p^N\in X} f(a)\mu_{1,\alpha}(a + p^N)\end{aligned}$$

letting N tend to ∞ gives the result. ■

We can choose $X = \mathbb{Z}_p^\times$ where $\mathbb{Z}_p^\times$ is the subset of $\mathbb{Z}_p$ such that for elements $x \in \mathbb{Z}_p^\times$, $|x|_p = 1$.

We note that we can let $g(x) = x^{k'-1}$ for for $k' \equiv k \pmod{(p-1)p^N}$, from which we can obtain

$$\left| \int_{\mathbb{Z}_p^\times} x^{k'-1} - \int_{\mathbb{Z}_p^\times} x^{k-1} \right|_p \le p^{-(N+1)}.$$

Then we can extend our function f from k to a continuous function of the p-adic integers; we have

$$\int_{\mathbb{Z}_p^\times} x^{s_0+(p-1)s-1}\mu_{1,\alpha}$$

for p-adic integers s. We will return to this expression later. Before that, we need to find a relationship between our p-adic ζ function and the expressions that we have found.

Proposition 3.6.

$$\zeta_p(1-k) = \frac{1}{a^{-k}-1} \int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}.$$

Proof. Recall that we showed

$$\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha} = \frac{1}{k}\int_{\mathbb{Z}_p^\times} \mathbb{1}_{\mu_{k,\alpha}}.$$

Evaluating the right hand side, we have

$$\begin{aligned}\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha} &= \frac{1}{k}\int_{\mathbb{Z}_p^\times} \mathbb{1}_{\mu_{k,\alpha}} \\ &= \frac{1}{k}\mu_{k,\alpha}\left(\mathbb{Z}_p^\times\right) \\ &= \frac{1}{k}(\alpha^{-k}-1)(1-p^{k-1})\left(-\frac{1}{k}\int_{\mathbb{Z}_p} \mathbb{1}_{\mu_{B,k}}\right).\end{aligned}$$

Interpolating the expression above gives the result. It is important to note here that we are working with $1-k$ instead of k, which is why the p^{-k} coefficient is replaced by p^{k-1}. ■

Using this representation, it is possible to derive an interesting property of the Bernoulli numbers.

Theorem 3.7 (Clausen and von Staudt)**.** *Assume that p is such that $p-1 \equiv 0 \pmod{k}$. Then*

$$pB_k \equiv -1 \pmod{p}.$$

Proof. Let $\alpha = p+1$. Note that our condition guarantees that $p > 2$, so we have

$$pB_k = -kp\left(-\frac{B_k}{k}\right) = \frac{-kp}{\alpha^{-k}-1}(1-p^{k-1})^{-1}\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}.$$

Letting $z = \operatorname{ord}_p k$, we find

$$\alpha^{-k}-1 = (1+p)^{-k}-1 \equiv -kp \pmod{p^{z+2}},$$

so

$$\frac{-kp}{\alpha^{-k}-1} \equiv 1 \pmod{p}.$$

Since $k \geq 2$ we must have $(1-p^{k-1}) \equiv 1 \pmod{p}$, so

$$\begin{aligned}pB_k &\equiv \int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha} \pmod{p} \\ &\equiv \int_{\mathbb{Z}_p^\times} x^{-1}\mu_{1,\alpha} \pmod{p} \\ &\equiv -1\end{aligned}$$

as desired. ■

Definition 3.8. Let $s_0 = \{0, 1, 2, \dots, p-2\}$. We let

$$\zeta_{p,s_0}(x) = \frac{1}{\alpha^{-s_0+(p-1)s}-1}\int_{\mathbb{Z}_p^\times} x^{s_0+(p-1)s-1}\mu_{1,\alpha}.$$

Note that if $k = s_0 + (p-1)k_0$, we have

$$\zeta_p(1-k) = \zeta_{p,s_0}.$$

However, note that ζ_{p,s_0} has arguments which are all contained in the p-adic integers.

It remains to prove that our function is solely dependent on p and s, so we need to prove that α is independent of ζ_{p,s_0}.

Theorem 3.9. *Assume α and ζ_{p,s_0} are as defined in Proposition 3.8. Then for fixed p and s_0, ζ_{p,s_0} does not depend on the choice of $\alpha \in \mathbb{Z}$ such that $\alpha \not\equiv 0 \pmod p$ and $\alpha \neq 1$.* [Kob77]

Proof. Clearly the integral is a continuous function in s. If when $s_0 = 0$, $s \neq 0$, we find that $\frac{1}{\alpha^{-(s_0+(p-1)s)}-1}$ is also a continuous function. Then ζ_{p,s_0} is also a continuous function. Then we just need to show that ζ_{p,s_0} is independent of α. Assuming ζ'_{p,s_0} is a function similarly defined, with a variable β instead of α, we find that when $k = s_0 + (p-1)s > 0$ we have $\zeta'_{p,s_0} = \zeta_{p,s_0} = (1-p^{k-1})\frac{-B_k}{k}$. Since nonnegative integers are dense in $\mathbb{Z}_p$, any continuous functions which are equal in the nonnegative integers are also equal. Thus ζ_{p,s_0} is independent of α. ■

4. Conclusion

We looked at the p-adic interpolation of the Riemann ζ-function, which translates the language of analytic number theory into the p-adic numbers. Iwasawa theory considers the properties of these p-adic functions and their relations to analytic functions in $\mathbb{C}$. Iwasawa theory is discussed further in [Was97].

References

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