Profinite Fibonacci Numbers

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Abstract

In this paper, we focus on profinite Fibonacci numbers, looking for values of n such that the nth number of the Fibonacci sequence is n. We find the 11 such fixed points of the sequence.

1 Introduction

You may be excited when you find 3 solutions to $F_n = n$ just by glancing over the initial terms of the Fibonacci sequence in $\mathbf{Z}$. After all, $F_0 = 0, F_1 = 1$, and $F_5 = 5$. But your happiness is short-lived since the other 8 nontrivial points lie somewhere else—these are the primary focus of this paper. Recall that F admits a unique continuous extension $\hat{\mathbf{Z}} \to \hat{\mathbf{Z}}$, called the Fibonacci map—it is here where we'll find our points.

Recall that for two numbers to be equal in $\hat{\mathbf{Z}}$, they must be congruent mod a large enough k. In other words, we know that $F_s \equiv s \pmod{n!}$ where $s = (...s_4s_3s_2s_1)_!$, the factorial base representation that you're no doubt familiar with.

We introduce a sister sequence of the Fibonacci numbers: the Lucas numbers, recursively defined as $L_n = L_{n-1} + L_{n-2}$ with starting values $L_0 = 2$ and $L_1 = 1$. Indeed they share many properties with the Fibonacci sequence.

We now indulge ourselves with a well-known identity of the Fibonacci sequence, where ϕ represents $\frac{1+\sqrt{5}}{2}$ and α represents $\frac{1-\sqrt{5}}{2}$, roots of $x^2 - x - 1$. Then,

$$F_n = \frac{\phi^n - \alpha^n}{\phi - \alpha}.$$

It can be proved rather easily using induction.

2 Needed Preliminaries

Using the representation of F_n in terms of ϕ and α and other logarithmic representations, we have

$$\frac{F_{n!}}{n!} = \frac{\phi^{n!} - \alpha^{n!}}{n!(\phi - \alpha)} = \frac{1}{\phi - \alpha}(\frac{\phi^{n!} - 1}{n!} - \frac{\alpha^{n!} - 1}{n!}).$$

Thus, taking the limit,

$$\lim_{n\to\infty}\frac{F_{n!}}{n!}=\frac{1}{\phi-\alpha}(log(\phi)-log(\alpha)).$$

Since $log(\phi)+log(\alpha)=log(\phi\alpha)=log(-1)=0$, we know that $log(\phi)=-log(\alpha)$. Then we can create the following definition.

Definition 2.1. We define l as:

$$l=\frac{log(\phi)}{\phi-\alpha}=\lim_{n\to\infty}\frac{F_{n!}}{n!}.$$

We'll use l continuously throughout the paper.

The following lemma plays an integral role in the ultimate theorem, so we introduce it, though its proof requires some previous number-theoretic results, which we do not wish to include for fear of distracting the reader from the main discussion.

Lemma 2.1. *If $n\equiv 1$ mod 5, then $gcd(n+1,1-L_{s_n}\frac{F_{n!}}{2n!})=5^k$ for some integer k.*

Another beautiful property is that we can create a power series expansion of F_s around s_0, detailed by the following lemma.

Lemma 2.2. *If $s_n\in\hat{\mathbf{Z}}$ and l is as defined earlier, then*

$$F_{s_n}\equiv\sum_{t\geq 0}5^t l^{2t+1}L_{s_0}\frac{(s-s_n)^{2t+1}}{(2t+1)!}+5^t l^{2t}F_{s_n}\frac{(s-s_n)^{2t}}{(2t)!}\pmod{m}.$$

This final lemma is perhaps the most important of them all, restricting the values we must consider when evaluating potential solutions to $F_n=n$ that are profinite integers, which inconveniently extend indefinitely to the left.

Lemma 2.3. *If we let a nonzero s be a fixed point of F, then*

$$v_5(1-lL_s)=j,$$

for some $1\leq j\leq 3$.

Its proof mainly makes use of the previous two lemmas, Lemma 2.1 and 2.2.

3 The 11 Points

Now we prove a theorem that we regard as the centerpiece of our exploration, from which the identities of the fixed points will directly sprout.

Theorem 3.1. *Let $n \geq 5$ be an integer and $1 \geq j \geq 3$ be the integer satisfying $j = v_5(1 - lL_{s_n})$. Then,*

$$\frac{F_{s_n} - s_n}{5^j n!} \equiv k\frac{1 - lL_{s_n}}{5^j} \pmod{n+1},$$

for all $n \geq 5j$. Furthermore, if we take $n \geq 15$, then k is uniquely determined by the congruence.

Proof. Recall that we need to find such a k here such that $F_{s_{n+1}} \equiv s_{n+1} \bmod n!$; we instead impose a stricter congruence $F_{s_{n+1}} \equiv s_{n+1} \pmod{5^j n!}$. Now we check the conditions for Lemma 2.2, which are indeed satisfied. Therefore, looking at the power series of F,

$$F_{s_n} \equiv \sum_{t\geq 0} 5^t l^{2t+1} L_{s_n} \frac{(kn!)^{2n+1}}{(2t+1)!} + 5^t l^{2t} F_{s_n} \frac{(kn!)^{2t}}{(2t)!} \pmod{5^j(n+1)!}.$$

When $j = 3$, it's apparent that all terms beyond the third disappear, so the same applies for $j < 3$. Thus, we can rewrite this congruence more simply as:

$$F_{s_{n+1}} \equiv lL_{s_n} kn! + F_{s_n} + C + D \pmod{5^j(n+1)!},$$

where

$$C = 5l^3 L_{s_n} \frac{(kn!)^3}{6} + 5l^2 F_{s_n} \frac{(kn!)^2}{2}, D = 25l^5 L_{s_n} \frac{(kn!)^5}{120} + 25l^4 F_{s_n} \frac{(kn!)^4}{24}.$$

Notice that $5l(kn!)^2$ divides both C and D and after some calculation that $5l(kn!)^2 \equiv 0 \pmod{5^j(n+1)!}$ if $n \geq 5j$. Then, we can further simplify our congruence to:

$$F_{s_{n+1}} \equiv lL_{s_n} kn! + F_{s_n} \pmod{5^j(n+1)!}$$

of course only when $n \geq 5j$. Now we'll use substitutions to prove the theorem. Just recall that $F_{s_{n+1}} \equiv s_{n+1} \equiv kn! + s_n \pmod{5^j(n+1)!}$,. Then, combining with our simplified congruence, we have

$$\frac{1 - lL_{s_n}}{5^j} kn! \equiv \frac{kn! + F_{s_n} - F_{s_{n-1}}}{5^j} \equiv \frac{F_{s_n} - s_n}{5^j} \pmod{(n+1)!}.$$

We divide everything by $n!$, and we're done. By Lemma 2.1, $\gcd(1-lL_{s_n}, n+1)$ is at least a power of 5 and at most a power of 5^j. Thus, for all $n \geq 15 \geq 5j$,

$$gcd(\frac{1 - lL_{s_n}}{5^j}, n+1) = 1.$$

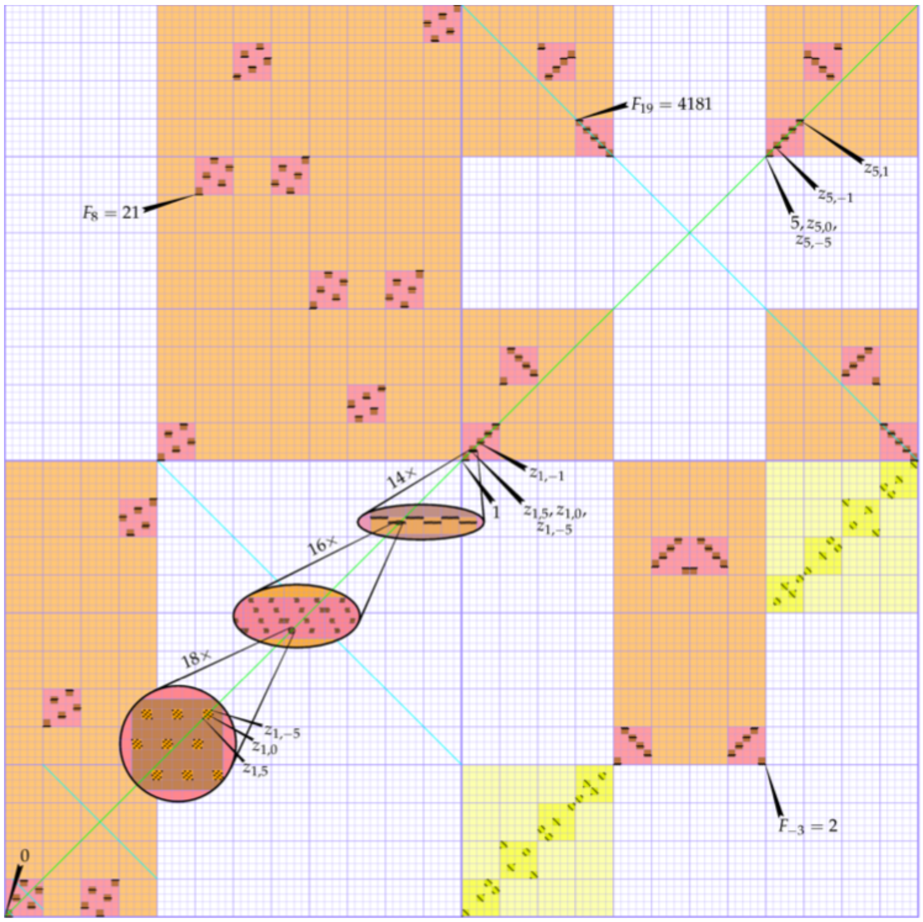
We've proved the uniqueness portion of the theorem i.e. k is uniquely determined by the congruence after the 14th digit. □

So we only need worry about the first 14 digits of the fixed points, and how many ways we can create numbers that satisfy all the conditions. Combining with the uniqueness of k when n does not leave a remainder of 4 (mod 5), we only consider the 4th, 9th, and 14th digits, each of which admits 5 possibilites for k that satisfies the congruence. Computationally, then these are the only 11 possibilities.

$$z_1 = (..., 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)_!,$$
$$z_2 = (..., 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)_!,$$
$$z_3 = (..., 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 1)_!,$$
$$z_4 = (..., 11, 2, 9, 0, 10, 0, 7, 1, 4, 1, 1, 0, 0, 1)_!,$$
$$z_5 = (..., 8, 11, 1, 3, 3, 4, 7, 1, 4, 1, 1, 0, 0, 1)_!,$$
$$z_6 = (..., 6, 5, 6, 5, 7, 8, 7, 1, 4, 1, 1, 0, 0, 1)_!,$$
$$z_7 = (..., 8, 0, 7, 3, 3, 9, 5, 3, 1, 2, 2, 0, 0, 1)_!,$$
$$z_8 = (..., 12, 8, 5, 2, 4, 4, 0, 0, 0, 0, 0, 0, 2, 1)_!,$$
$$z_9 = (..., 10, 2, 10, 4, 8, 8, 0, 0, 0, 0, 0, 0, 2, 1)_!,$$
$$z_{10} = (..., 11, 11, 11, 2, 4, 8, 7, 1, 4, 1, 1, 0, 2, 1)_!,$$
$$z_{11} = (..., 3, 11, 3, 11, 0, 9, 1, 6, 2, 4, 4, 0, 2, 1)_!.$$

Although there are several other profinite solutions to $F_s \equiv s \pmod{15!}$, all of them violate some component of Theorem 3.1, as we can computationally verify.

We conclude our exploration into profinite Fibonacci numbers by offering a striking graphic just for recreation, created by Willem Jan Palenstijn. The Fibonacci function $\hat{\mathbf{Z}} \to \hat{\mathbf{Z}}$ is on display, where every profinite integer of the form $(...c_3c_2c_1) = \sum_{i\geq 1} c_i i!$ is represented as $\sum_{i\geq 1} \frac{c_i}{(i+1)!}$. Therefore, rather cleverly, we can show the entirety of the graph $\{(s, F_s) : s \in \hat{\mathbf{Z}}\}$ just within a unit square. Then, the points that intersect with the $y = x$ diagonal, green in color, are special: indeed they are the 11 values of n where $F_n = n$.



4 References

We simply reworked the important proofs, but much of the intellectual genius we found in these resources written by these fabulous people, to whom we express all our gratitude.

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