Cantor Set and the p-adics

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1 Definitions

This expository paper will begin with showing that the Cantor Set and the p-adics, specifically the p-adic integers are homeomorphic. Then it will go on to show that removing one point from it gives the p-adic numbers, i.e. $\mathbb{Q}_p$.

Definition 1 *The Cantor Set is defined as the set of points remaining after starting with the interval* $[0, 1]$*, and at each iteration removing the middle third open interval from every remaining interval.*

Formally, let $C_0 = [0, 1]$*. Then* C_n *is the set remaining after the middle thirds are removed from* C_{n-1}*. The Cantor Set is defined as*

$$C = \bigcap_{n=0}^{\infty} C_n \tag{1}$$

.

Definition 2 *Let* p *be any prime number. A generalized variant* $C^{(p)}$ *is defined as the set of points remaining after starting with the interval* $[0, 1]$ *and at each iteration dividing every interval into* $2p - 1$ *sub-intervals and removing alternating open intervals.*

Again, let $C_0^{(p)} = [0, 1]$*. Then* $C_n^{(p)}$ *is the set remaining after the alternating open intervals are removed from* $C_{n-1}^{(p)}$*. Define*

$$C^{(p)} = \bigcap_{n=0}^{\infty} C_n^{(p)} \tag{2}$$

.

2 Cantor Set and $\mathbb{Z}_p$

The clearest homeomorphism is between the p-adic integers for an arbitrary p, and its corresponding Cantor Set variant.

An easy example to see is the function on $\mathbb{Z}_2 \to C^{(2)}$ which is defined as $\sum_{n=0}^{\infty} a_n 2^n \to \sum_{n=0}^{\infty}(2a_n)3^{-(n+1)}$.

Observe that the digit expansion for $\mathbb{Z}_2$ is $\ldots x_3x_2x_1x_0$ in base 2, where every $x_i \in \{0,1\}$, while the digit expansion for $C^{(2)}$has base 3 expansion $x_0.x_1x_2x_{3\ldots}$ where every $x_i \in \{0,2\}$. Though the digits in the two types of numbers run in opposite directions, there is a clear a bijection between corresponding digits from each.

The fact that every digit in Cantor set is 0 or 2 requires justification.

Lemma 3 *$C^{(2)}$ consists precisely of the real numbers in $[0,1]$ whose base 3 expansions only contain the digits 0 and 2.*

Proof. Suppose $x \in C^{(2)}$. Then x is an element in C_n for all $n \geq 0$. The numbers in each C_n are exactly those whose n^{th} truncation uses contains only the digits 0 and 2 in base 3. The right endpoint of any interval will terminate on a singular 1, and any terminating 1 can be replaced by 0222... . As $n \to \infty$, every digit of x will use only 0 and 2 in its base-3 expansion.

Conversely, suppose $x \in [0,1]$ contains only the digits 0 and 2 in its base 3 expansion. Then its n^{th} truncation x_n must be the left endpoint of an interval in C_n. Since $x_n \leq x \leq x_n + \frac{1}{3^n}$, x must be contained in C_n for all $n \geq 0$. Thus $x \in C^2$. ■

A generalization is desired to show the same for any prime.

Lemma 4 *$C^{(p)}$ consists precisely of the real numbers in $[0,1]$ whose base p expansions only contain even digits between $[0, 2p-1]$.*

The proof for this is similar to the case for $p = 2$. Now we can move on to the generalized homeomorphism!

Theorem 5 *The Cantor Set $C^{(p)}$is homeomorphic to $\mathbb{Z}_p$*

Proof. Define a function $f : \mathbb{Z}_p \to C^{(p)}$,

$$\sum_{n=0}^{\infty} a_n p^n \to \sum_{n=0}^{\infty}(2a_n)(2p-1)^{-(n+1)} \tag{3}$$

Let $x = \sum_{n=0}^{\infty} x_n p^n$ and $y = \sum_{n=0}^{\infty} y_n p^n \in \mathbb{Z}_p$.

Then $|x - y| \leq p^{-k}$ for $k \geq n$, which implies that the first k digits of x and y are the same. This implies that the first k digits in $f(x)$ and $f(y)$ are equal. Thus, $|f(x) - f(y)| \leq (2p-1)^{-k}$. Therefore, f is a continuous mapping from $\mathbb{Z}_p$ to $C^{(p)}$, hence a homeomorphism. ■

There is another way to show homeomorphism between the Cantor set and $\mathbb{Z}_p$, by showing that its properties are unique to the Cantor set. If we can show that $\mathbb{Z}_p$ satisfies all of those properties, then it must be homeomorphic to $C^{(p)}$ because the Cantor set is the only space to do so.

Theorem 6 (Brouwer's Theorem) *A topological space is a Cantor space if and only if it is non-empty, perfect, compact, totally disconnected, and metrizable.*

Lemma 7 *The p-adic integers satisfy the following properties:*

1. $\mathbb{Z}_p$ *is nonempty*
2. $\mathbb{Z}_p$ *is totally disconnected*
3. $\mathbb{Z}_p$ *is perfect*
4. $\mathbb{Z}_p$ *is compact*
5. $\mathbb{Z}_p$ *is a metric space*

Brouwer's theorem tells us that the topological properties of $\mathbb{Z}_p$ make it homeomorphic to the Cantor set.

Furthermore, this universality allows us to show homeomorphism between all Cantor sets, and thus all p-adic sets of different primes. We have been using the term "Cantor set" to mean $C^{(2)}$ and $C^{(p)}$ interchangeably because all of the Cantor set variants are actually equivalent to *the* Cantor set—the original middle-thirds set.

As a nice aside, the homeomorphism also allows us to instantly register other properties of $\mathbb{Z}_p$ by bringing them over from C, such as the cardinality of $\mathbb{Z}_p$, which must be uncountable since C is an uncountable set. (Of course, there are proofs directly on the p-adics which give the same result)

3 Throwing out a point

If the p-adic integers are homeomorphic to the Cantor Set, then what does the full set of p-adic numbers look like?

Proposition 8 *The Cantor Set with one point removed is homeomorphic to* $\mathbb{Q}_p$.

The reasoning for this takes advantage of the fact that the Cantor set is a fractal. Starting with $\mathbb{Z}_p$ is homeomorphic to a Cantor set, it won't matter which Cantor set is used in the proof, so might as well use $C^{(2)}$, whose elements are represented with base 3 expansion.

All points in the Cantor set are identical, so it does not matter which one is removed.

From here we can construct the homeomorphism.

Proof. Take an infinite sequence of sets of p-adic numbers, $\mathbb{Z}_p, p^{-1}\mathbb{Z}_p, p^{-2}\mathbb{Z}_p, \ldots$ Each term in this sequence is the ring of p-adic integers divided by higher and higher powers of p.

Because the Cantor set is a fractal, it can be scaled, so $\mathbb{Z}_p$ is homeomorphic to half of the Cantor Set, on the closed interval $[0.2, 1]$. Call this C_0^*. Then set $p^{-1}\mathbb{Z}_p$ homeomorphic to C_1^* on $[0.02, 0.1]$. Continue on by setting the subset C_n^* on the interval $[2 \cdot 3^{-(n+1)}, 3^{-n}]$ to be homeomorphic to $p^{-n}\mathbb{Z}_p$.

Taking the infinite union of these subsets gives:

$$\bigcup_{n=0}^{\infty} C_n^* = C\backslash\{0\} \tag{4}$$

In a similar manner,

$$\bigcup_{n=0}^{\infty} p^{-n}\mathbb{Z}_p = \mathbb{Q}_n \tag{5}$$

And since every Cantor subset C_n^* is homeomorphic to $p^{-n}\mathbb{Z}_p$, the infinite union of all C_n^* is homeomorphic to the infinite union of $p^{-n}\mathbb{Z}_p$, which is $\mathbb{Q}_n$.

■

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