ON THE STRANGE 3-ADIC PROPERTY

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1. Introduction

As we learned such p-adic numbers, we focused largely on the general identities and properties of $\mathbb{Q}_p$ and $\mathbb{Z}_p$. However, if we take a look in a specific p, we may notice some strange identities. Here, we are going to observe when $p = 3$. This occurrence may seem familiar as the last problems of the chapter 9 of our textbook has the same question.

2. Problem

If k is a positive integer, let $3^{v(k)}$ be the highest power of 3 diving k.Put

$$r(n) = \sum_{i=0}^{n-1} \binom{2i}{i}$$

for positive integers n. Prove that

(1): $v(r(n)) \geq 2v(n)$

(2): $v(r(n)) = v(\binom{2n}{n}) + 2v(n)$

This seems quite complicated yet straightforward. First let's understand what is $\binom{2n}{n}$ is. In p-adic integers, binomials can be understood as

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$

Based on this understanding, now diving into the solution suggested by Nicholas Strauss and Jeffery Shallit.

3. Solution

First step is to restate the question. This proof can be changed into following form:

$$v(\sum_{k=0}^{n-1} \binom{2k}{k}) = v(n^2 \binom{2n}{n})$$

We state $v(\cdot)$ as 3-adic valuation. In other words, just like $|\cdot|_3$. From this statement, we can give a simple proof and branching out properties of $\mathbb{Z}_3$. Let set function f(x) as:

$$f(n) = \frac{\sum_{k=0}^{n-1}}{n^2\binom{2n}{n}}$$

With the statement, this function should mean that this function should be an unit, a solid number. With some calculations, we can reach to the simple observations that

$$f(n) \equiv -1 \pmod 3 \quad \forall n$$

and

$$n \equiv m \pmod{3^j} \implies f(n) \equiv f(m) \pmod{3^{j+1}}$$

Expanding this observation into theorem we can obtain the following:

Theorem 3.1. *The function f extends to a 3-adic analytic function from* $\mathbb{Z}_3$ *to* $-1 + 3\mathbb{Z}_3$. *Its values at negative integers and half-integers are rational numbers, given by*

$$f(-n) = -\frac{(2n-1)!}{n!^2}\sum_{k=0}^{n-1}\frac{k!^2}{(k=1)!} \quad (n \geq 1)$$

$$f(-n-\frac{1}{2}) = -\frac{2^{4n+2}}{(2n+1)^2\binom{2n}{n}}\sum_{k=0}^{n}2^{-4k}\binom{2k}{k}$$

As a corollary, we get the following identities,

$$v(\sum_{k=0}^{n-1}\frac{k!^2}{(2k+1)!}) = v(\frac{n!^2}{(2n-1)!}) \quad (n \geq 1)$$

$$v(\sum_{k=0}^{n}2^{-4k}\binom{2k}{k}) = v((2n+1)^2\binom{2n}{n}) \quad (n \geq 0)$$

Proof it following:

By observing the definition of f(x) above, we can rearrange it into:

$$(2n+1)(2n+2)f(n+1) = 1 + n^2 f(n)$$

for $n \in \mathbb{N}$.If f has ab extension to a 3-adic continuos function, $\mathbb{Z}_3\mathbb{Z}_3$, then this functional equation should also hold in $\mathbb{Z}_3$. Since the left-side vanish if $n = -1, -\frac{1}{2}$, we must have $f(1) = -1$ and $f(-\frac{1}{2}) = -4$. Further values are achievable through calculation based on the functional equation above and the induction. Therefore, the only task left is to proof the first statement of the theorem.

Let $h(x) = 2nf(n)$. It is possible to show such $h(x)$ extends to a 3-adic analytic function, and then $x|h(x)$. In order to use the recursion, the functional equation of $f(x)$ becomes,

$$2(2n+1)g(n+1) = 2 + ng(n)$$

Defining rational numbers a_n such that

$$g(n) = \sum_{r=0}^{\infty} a_r \binom{n-1}{r}$$

for $n \in \mathbb{N}$. If it is true that $v(a_r) \xrightarrow{\infty}$ as $r \to \infty$ then naturally $g(n)$ wull converge 3-adically for all $n \in \mathbb{Z}_3$ and show that $f(n)$ is a 3-adically analytic function. By substituting $g(n)$ formula into its functional eqaution,

$$2 + \sum_{r=0}^{n-1}(r+1)a_r\binom{n}{r+1} = \sum_{r=0}^{n}[2(2r+1)\binom{n}{r} + 4(r+1(\binom{n}{r+1})]a_r$$

By comparing the coefficients of the binomials $\binom{n}{r}$, we get, $2(2r+1)a_r = -3ra_{r-1}$ for all $r \geq 1$, which becomes,

$$a_r = \frac{(-3)^r r!^2}{(2r+1)!} \quad (r \geq 0)$$

The 3-adic valuation, $v(n)$, of this tend to approach ∞ with r, ($\because v(3^r/(2r+1)!) \geq 0$ and $v(r!) \to \infty$, which shows the analytic continuation of $g(x)$.
Before we finsihe the proof we need to make sure the following lemma is also true.

Lemma 3.2. *The series* $\sum_{r=0}^{\infty} \dfrac{3^r r!^2}{(2r+1)!}$ *converges 3-adically to 0.*

Assuming the lemma is true, we can change $g(n)$:

$$g(n) = \sum_{r=0}^{n-1}(-3)^r \frac{r!}{(2r+1)!}(n-1)(n-2)\cdots(n-4)$$

$$= \sum_{r=0}^{n-1} \frac{3^r r!2}{(2r+1)!} - \frac{1}{2}n + \sum_{r=2}^{n-1}(-3)^r \frac{r!}{(2r+1)!}[(n-1)(n-2)\cdots(n-r) - (-1)^r r!]$$

By the lemma, the first term should have valuation,

$$v(\sum_{r=0}^{n-1} \frac{3^r r!^2}{(2r+1)!}) = v(\sum_{r=n}^{\infty} \frac{3^r r!^2}{(2r+1)!} \geq 2\frac{n-2}{3}(n) + 1 \quad (n \geq 4)$$

$(\because v((3^r r!^2)/(2r+1)!) \geq 2v(r!) \geq 2(r-2)/3)$
Also, $(n-1)(n-2)\cdots(n-r) - (-1)^r r!$ is divisible by n and $(-3)^r r!/(2r+1)!$ is divisible by 3 for all $r \geq 2$, the transformed form of $g(n)$ gives

$$g(n) \equiv -\frac{1}{2}n \pmod{3^{v(n)+1}}$$

which implies that $f(n) = g(n)/2n$ is indeed 3-integral and congruent to $-1 \pmod 3$. Therefore our proof is complete.
Proof of Lemma is following:
Let's take a look at the following power series identity.

$$\sum_{r=0}^{\infty} \frac{r!^2}{(2r+1)!} = \sum_{r=0}^{\infty} (\int_0^1 t^r (1-t)^r dt) x^r$$

$$= \int_0^1 \frac{dt}{1+xt+xt^2} = \frac{1}{\sqrt{x^2-4x}} \log \frac{2-x+\sqrt{x^2-4x}}{2-x-\sqrt{x^2-4x}}$$

$$= \frac{1}{3\sqrt{x^2-4x}} \log \frac{2-x+\sqrt{x^2-4x}/4}{2-x-\sqrt{x^2-4x}/4}$$

$$= \frac{1}{3\sqrt{x^2-4x}} \log \frac{2-x(3-x)^2+(3-x)(1-x)\sqrt{x^2-4x}}{2-x(3-x)^2-(3-x)(1-x)\sqrt{x^2-4x}}$$

$$= \frac{2}{3} \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{x^n (x-4)^n (3-x)^{2n+1} (1-x)^{2n+1}}{(2-x(3-x)^2)^{2n+1}}$$

in $\mathbb{Q}[[x]]$. Since both sides converges to 0 and right hand side vanishes as $x = 3$, we can easilly end the lemma.
Since now we know that the $f(x)$ is a 3-adically analytic function, we can easily see how the 2 identities we wanted to show are obviously true after all $(\because v(\binom{2n}{n}) \geq 0)$.

4. **Conjecture**

The series $\sum_{r=0}^{\infty} \frac{3^r r!^2}{(2r+1)!} \sigma(2)(1, \frac{1}{2}, \cdots, \frac{1}{r}$ converges 3-adically to 0.