MONSKY'S THEOREM

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In this paper, we will prove Monsky's Theorem:

Theorem 0.1. *It is not possible to dissect a triangle into an odd number of triangles of equal area.*

In order to prove Monsky's Theorem, we will need Sperner's Lemma and 2-adic numbers.

1. Sperner's Lemma

First, we discuss Sperner's Lemma:

Lemma 1.1. *Let T be a triangle. T is triangulated (divided into smaller triangles that meet edge-to-edge) and each vertex of the small triangles are colored such that:*

1. One vertex of T is green, another is red, and the third is blue.

2. The side containing the green and red endpoints contains only red and green points, and vice versa for the other sides.

Then, there exists an odd number of triangles that have a vertex of each color. This also means there is at least one triangle with this property.

To prove this lemma, we start by showing that there are an odd number of line segments colored red at one endpoint and green at the other on the edge with endpoints that are red and green. Let this edge be the bottom one, with the endpoint on the right colored red and the endpoint on the left colored green.

We now label the line segments in this edge by the colors of the endpoints. If the colors of the endpoints of the segment are both red or both green, then we label the segment 0. On the other hand, if the right endpoint is green and the left is red, we label it -1, and if the right endpoint is red and the left is green, we label it 1. We then notice that the sum of these numbers will always equal 1; therefore, there are an odd number of segments that are labeled 1 or -1.

We now think of the red-and-green segments as doors, and all other segments as walls. If we want to trace a path through the triangle starting from a door on the bottom edge, we notice that it will either end at a triangle with a vertex of each color, since two out of three of the sides will be walls, or it will exit the triangle. For every door on the bottom edge, there is a unique path, because there will never be a triangle where all three sides are doors.

Since a path that enters the triangle through a door and exits it again will include an even number of doors, there must be a path that ends at a triangle with a vertex of each color. Therefore, there is at least one such triangle.

2. Proving Monsky's Theorem with 2-Adic Numbers

We now use Sperner's Lemma and 2-adic numbers to prove Monsky's Theorem.

Without loss of generality, let the square being dissected have vertices at $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. To start, we label every point in the plane a certain color according to their 2-adic valuations. This requires extending the p-adic valuation from the rationals to the reals, so we show that this is possible using the following theorem, which we will not prove here:

Theorem 2.1. *Let $(K, |\cdot|)$ be a normed field and L/K be an extension field. If $(K, |\cdot|)$ is non-Archimedean, then there is a norm on L extending $|\cdot|$.*

We also define what it means to be non-Archimedean:

Definition 2.1. *An absolute value on K is non-Archimedean if for any x, y in K,*

$$|x+y| \leq \max(|x|, |y|).$$

This is true of the 2-adic absolute value, so by Theorem 2.1, there is an extension of the 2-adic valuation to the reals. We now color the points in the plane as such:

If $|x| < 1$ and $|y| < 1$, then we color (x, y) red.

If $|x| \geq 1$ and $|x| \geq |y|$, then we color (x, y) blue.

If $|y| \geq 1$ and $|y| > |x|$, then we color (x, y) green.

For example, the point $(0, 0)$ would be labeled red, $(0, 1)$ would be green, and the points $(1, 0)$ and $(1, 1)$ would be blue. We notice that since points on the side with endpoints $(0, 0)$ and $(0, 1)$ lie on the line $x = 0$, they are labeled either red or green. Similarly, the points on the side with endpoints $(0, 0)$ and $(1, 0)$ are labeled red or blue, the points on the side with endpoints $(0, 1)$ and $(1, 1)$ are labeled green or blue, and the points on the side with endpoints $(1, 0)$ and $(1, 1)$ are also labeled green or blue.

Since the two sides containing points labeled blue or green are adjacent, we can think of this square as a triangle that is bent at one side. Since Sperner's Lemma applies to this triangle, it must also apply to this square, so there is at least one triangle with vertices of all three colors.

Let the red vertex of this triangle be $A(a_1, a_2)$, the blue vertex be $B(b_1, b_2)$, and the green vertex be $C(c_1, c_2)$.

First, we show that if we shift the triangle so that A is at the origin, the vertices will not change colors. A will clearly stay red, since $(0, 0)$ is labeled red. The x coordinate of B is now $b_1 - a_1$. $|\cdot|_2$ satisfies the property that $|x - y| = \max(|x|, |y|)$ unless $|x| = |y|$. We know that $|b_1| \neq |a_1|$ since $|a_1| < 1$ and $|b_1| \geq 1$, so $|b_1 - a_1| = \max(|b_1|, |a_1|) = |b_1|$, which is greater than 1. Additionally, the y coordinate of B is now $b_2 - a_2$. $|b_2 - a_2| \leq \max(|a_2|, |b_2|)$, which will be less than $|b_1|$, so the second property of a blue point holds, and B is still blue.

Similarly, the y coordinate of C is now $c_2 - a_2$. Since $|c_2 - a_2| = \max(|c_2|, |a_2|) = |c_2|$, since $c_2 \geq 1$, while $a_2 < 1$. The x coordinate of C is $c_1 - a_1$, and $|c_1 - a_1| \leq \max(|c_1|, |a_1|)$, which will be less than c_2. Thus C stays green.

Let the new coordinates of B be $(b_3, b_4$, and the new coordinates of C be c_3, c_4. We can write the area of triangle of triangle T as a determinant:

$$\frac{1}{2}\begin{vmatrix} b_3 & b_4 \\ c_3 & c_4 \end{vmatrix} = \frac{1}{2}(b_3c_4 - b_4c_3)$$

One of the properties of the absolute value is that for any x, y, $|xy| = |x||y|$, so taking the absolute value of the area of T gives us:

$$\begin{aligned} |\tfrac{1}{2}(b_3c_4 - b_4c_3)| &= 2(|(b_3c_4 - b_4c_3)| \\ &= 2 * \max(|b_3c_4|, |b_4c_3|) \\ &= 2|b_3||c_4| \end{aligned}$$

Thus, the absolute value of the area of T is greater than 1. If the square is dissected into n triangles of equal area, then $nA = 1$. Taking the absolute value, $|nA| = |n||A| = 1$. $|A| > 1$, so $|n| < 1$, which means that n must be even.