Monsky's Theorem

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1 Introduction

Theorem 1.1. *It is not possible to dissect a square into an odd number of triangles with equal areas.*

One result we will use to prove this theorem is Sperner's lemma.

Lemma 1.2 (Sperner's Lemma)**.** *Every Sperner coloring of a triangulation of an n-dimensional simplex contains a cell colored with a complete set of colors.*

Note that a simplex is a generalization of the notion of a triangle or tetrahedron to a random dimension.

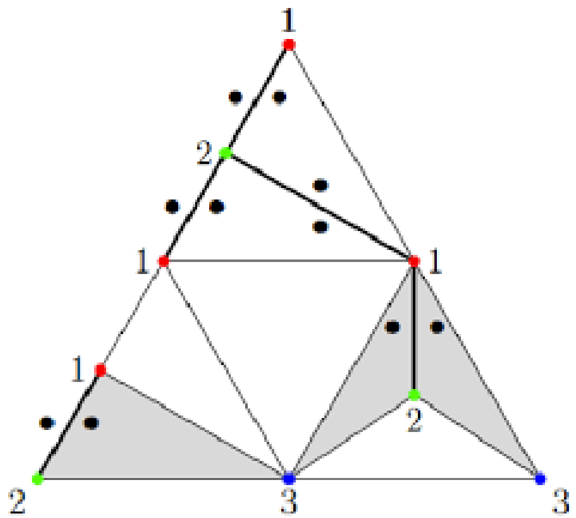

Figure 1: Depiction of the method of the proof of Sperner's lemma.

Proof. We present a case analysis.

- Case:1-dimensional. There is a line segment (a, b) divided into smaller segments, and we color the vertices of each subdivision with one of 2 colors. It is required that a and b receive different colors. Thus we must switch color an odd number of times, so that we get a different color for b. As a result there is an odd number of small segments that receive two different colors.
- Case:2-dimensional. (note that 1, 2, and 3 are different colors) Colored simplicial subdivision of a triangle T. Let Q denote the number of cells colored $(1, 1, 2)$ or $(1, 2, 2)$, and R the number of multi cells, colored $(1, 2, 3)$. Consider edges in the subdivision whose endpoints receive colors 1 and 2. Let X denote the number of boundary edges colored $(1, 2)$, and Y the number of interior edges colored $(1, 2)$ (inside the triangle T). Now count in 2 ways -
  1. Cells of the subdivision: For each cell of type Q, we get 2 edges colored $(1, 2)$, while for each cell of type R, we get exactly 1 such edge. Using this logic we also know that the number of interior edges (Y) $(1, 2)$ are counted twice and the boundary edges (X) $(1, 2)$ are only counted once. Thus we have that $2Q + R = X + 2Y$.

2. Boundary of T: Edges colored $(1, 2)$ can be only inside the edge between two vertices of T colored 1 and 2. As a result of the 1-dimensional case, between 1 and 2 there must be an odd number of edges colored $(1, 2)$. Hence, X is odd. This implies that R is also odd based upon our previous conclusion.

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