Euler Circle Writeup

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The Fibonacci numbers are a familiar sequence of integers defined by the recurrence $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers as stated are only defined for the ordinary nonnegative integers (and easily extendible to the negatives,) but it's also possible to extend the definition naturally to the profinite integers.

We went over the profinite integers in class, but it may be helpful to redefine them anyways. Profinite integers are members of the ring $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, or, equivalently, leftwards-infinite 'factorial base' expansions, in which the k-th term is multiplied by $k!$. These two expressions are equivalent because sufficiently large factorials are clearly divisible by any power of any prime p, and (by the CRT) knowing the p-adic part puts no constraints on the q-adic part for any other prime q.

The reason that the Fibonacci numbers extend so nicely is that they're periodic mod n for all n. This is fairly easy to see, there are only finitely many possible states of the recurrence mod n, and it's reversible, so iterating it must result in a loop from any starting point. This means that, given a profinite number x, F_x mod n is uniquely determined, by x mod some other number. This, for all n, is sufficient to uniquely specify F_x.

This extension has a number of interesting properties. It's continuous (which can be seen by looking modulo large factorials,) and satisfies the normal Fibonacci recurrence, since it's satisfied mod n for all n. A number of similar properties of the Fibonacci numbers probably also hold, I haven't checked this but I'm pretty sure you could use the exact same method.

One other interesting aspect of the extension is the existence of fixed points beyond the familiar 0, 1, and 5. There are 11 (including these) in total, determined by their values mod 5^n and 6^n. These fixed points often nearly satisfy some remarkable identities, in the most extreme example, the square of one (corresponding to values of 5 and -5 only differs from 25 at the 400th digit. I'm not entirely sure why this happen, but it's really neat.