# EUCLIDEAN MODELS OF $\mathbb{Z}_p$ AND $\mathbb{Q}_p$

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## 1. Introduction

**Definition 1.1.** [6] Two objects are said to be homeomorphic if they can be deformed into each other by a continuous, invertible mapping. (Rather, continuous and one-to-one in surjection, and having a continuous inverse.)

*Example.* A doughnut and a coffee cup are homeomorphic topologically.

**Definition 1.2.** [4] A subset $E \subseteq \mathbb{R}^n$ which is homeomorphic to $\mathbb{Z}_p$ is called a Euclidean model of $\mathbb{Z}_p$.

When $n = 1$, $E$ is called a linear model of $\mathbb{Z}_p$. If $\mathbb{R} > 1$, $E$ is a nonlinear model of $\mathbb{Z}_p$.

## 2. Linear Models of $\mathbb{Z}_p$

2.1. **Cantor Sets [4] [2].** The classical Cantor Set $C$ is the set obtained by taking the interval $[0, 1]$ in $\mathbb{R}$, and removing the middle third, leaving you with $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then, remove the middle third of each of those segments, leaving you with $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continue removing the middle third of each segment, ad infinitum.

There are some interesting properties of the Cantor Set. For example:

(1) $C$ is totally disconnected
(2) $C$ is uncountable [1]
(3) $0 \neq C$ is compact

**Definition 2.1.** A space $X$ is said to be totally disconnected if for every two distinct points $y, z \in X$, there are open subsets $Y, Z \subset X$ with $Y \cup Z = X$, $y \in Y$, $z \in Z$, and $Y \cap Z = 0$.

Another way of thinking about this definition is that a set is totally disconnected any two points can be separated by a decomposition of the original set - there are no nontrivial connected sets.

**Definition 2.2.** A subset $A$ of a metric space $X$ is said to be compact if every open cover has a finite subcover. Another way of saying this is it contains all its limit points and is bounded.

**Definition 2.3.** A set $X$ is nowhere dense if the union of all the open subsets of the smallest closed set containing $X$ is the empty set. [3]

One way of thinking about this is to say that if $A$ has nonempty intersection with nonempty set $U$, then we can "shrink" $U$ to a nonempty open set $V$ which is disjoint from $A$ .

**Definition 2.4.** A set $P$ is perfect if it is closed, and contains no isolated points. Rather, in a perfect set, any point can be approximated arbitrarily well by other points in the set.

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The classical cantor set is a Euclidean model of $\mathbb{Z}_p$. How so? Let us use the example of the ternary cantor set with $\mathbb{Z}_2$. The $n$th iteration of removing the middle section of each segment breaks the original segment into two portions, which we can say represent the residue classes modulo $2^n$. For example, after the first iteration, we have two segments, representing the numbers that are 0 (mod 2) on the left and 1 (mod 2) on the right. The second iteration breaks the left segment into a section representing numbers that are 0 (mod 4) and 2 (mod 4), and the right segment into sections representing the numbers that are 1 (mod 4) and 3 (mod 4), from left to right.

As it turns out, any linear model of $\mathbb{Z}_p$ is homeomorphic to the cantor set. In fact, more generally, any perfect, compact, totally disconnected metric space is homeomorphic to the Cantor set. [5]

*Proof.* By the definition of a totally disconnected metric space, any point is contained within a set of arbitrarily small diameter which is both closed and open. So, consider a cover of the space by such sets, of diameter $\leq 1$. It is compact, so we can take a finite subcover. Since any finite intersection of such sets is still both closed and open, by taking all possible intersections we will obtain a partition of the space into finitely many closed and open sets of diameter $\leq 1$. Since the space is perfect, no element of this partition is just a point, so, a further division is possible. Repeating this procedure for each set in the cover, by covering it by sets of diameter $\leq 1/2$, we obtain a finer partition. Proceeding by induction, we obtain a nested sequence of finite partitions into closed and open sets of positive diameter $\leq 1/2n, n = 0, 1, 2, \ldots.$ By mapping elements of each partition inside a nested sequence of contracting intervals, we construct a homeomorphism of the space onto a nowhere dense perfect subset of $[0, 1]$. Since any compact, perfect, totally disconnected subset of the real line is homeomorphic to the Cantor set (see [5] for the proof of this) our space is homeomorphic to the Cantor set. ■

Since $\mathbb{Z}_p$ is compact, perfect, and totally disconnected, any linear model of $\mathbb{Z}_p$ must also have these properties. Therefore, any linear model of $\mathbb{Z}_p$ must be homeomorphic to the Cantor set.

## 3. Nonlinear Models of $\mathbb{Z}_p$

One notable example of a nonlinear model of $\mathbb{Z}_p$ is of course Sierpinski's Triangle (Figure 1). It's fairly easy to see how this is a model of $\mathbb{Z}_3$. The top triangle represents 0 mod 3, the left represents 1 mod 3, and the right represents 2 mod 3, and so it continues similar to the cantor set.

Another nonlinear model of $\mathbb{Z}_p$ which we explored in week 2 is simply circles inside of circles (Figure 4). As usual, each circle represents some residue class modulo $p^n$ for some $n$. Note also that in this diagram, open balls are actually circles. Inside the biggest circle we have $B_0(\frac{1}{3}), B_3(\frac{1}{3})$, and $B_6(\frac{1}{3})$

There are also other fractal models of $\mathbb{Z}_p$. Figure 2 is a great example, with each arm and the center being a residue class modulo 5, and so on.

## 4. Euclidean Models of $\mathbb{Q}_p$

Models of $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are very similar, except for one main difference. Models of $\mathbb{Q}_p$ are infinitely large. If $\mathbb{Z}_p$ is a Sierpinski's Triangle, $\mathbb{Q}_p$ is 3 Sierpinski's Triangles stacked on top of each other, and 3 of those stacked on top of each other, etc. To think about it a little differently, the three triangles inside $\mathbb{Z}_p$ are 3-adic numbers which end in 0, 1, or 2. Inside

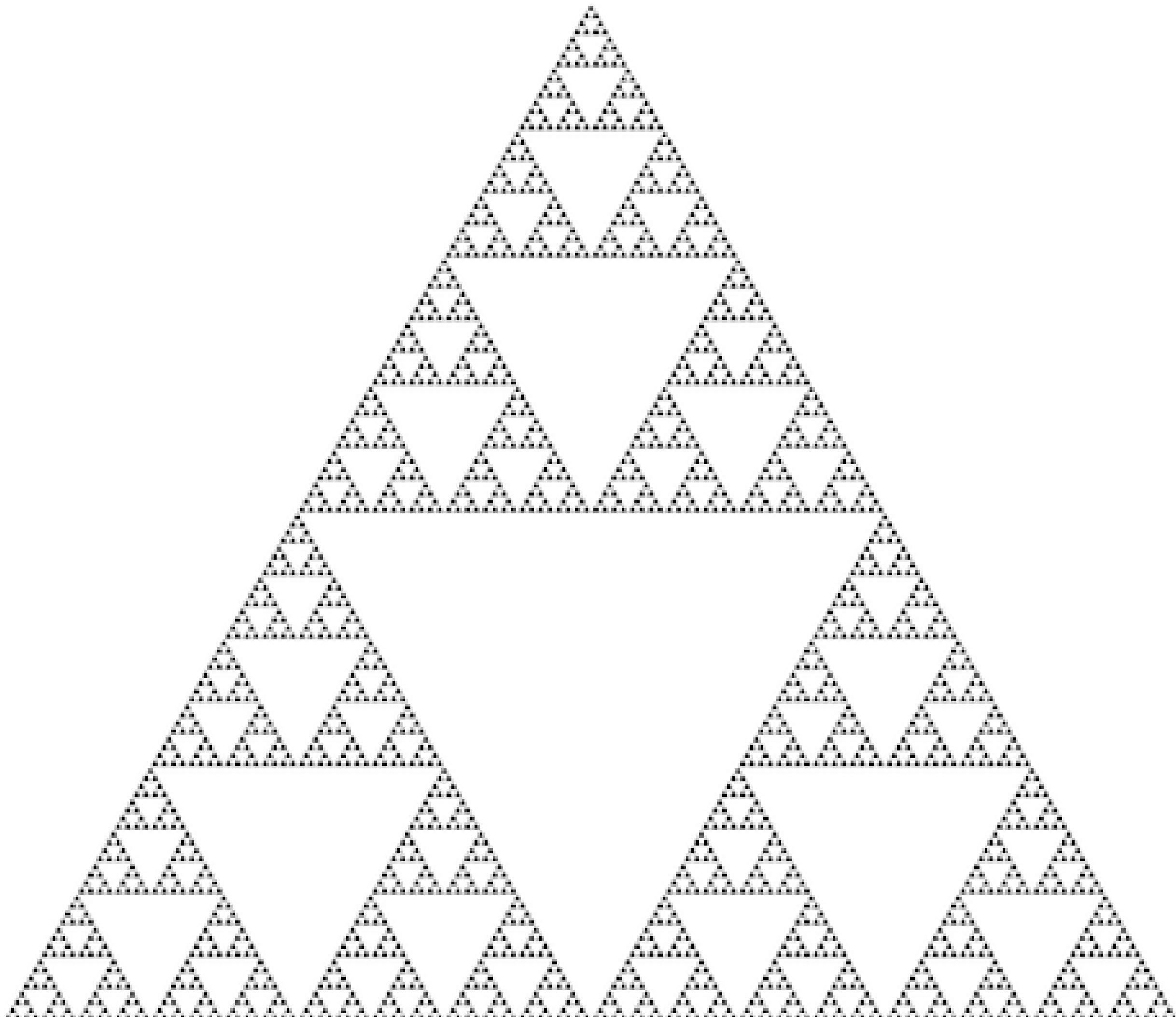

**Figure 1.** Sierpinski's Triangle

the 2 triangle are numbers which end in 02, 12, or 22, etc. When we expand beyond $\mathbb{Z}_p$, all we get $\mathbb{Z}_p$ being the ones which end in .0, and the other three triangles the same size as being the numbers which end in .1 or .2

## References


[1] Shishir Agrawal. "Cantor Set". In: (July 2015). URL: `https://math.berkeley.edu/~sagrawal/teaching/su15_math104/lec8_cantor.pdf` (visited on 06/04/2018).

[2] Zhixing Guo. "Cantor Set and Its Properties". In: (Apr. 2014). URL: `http://web.math.ucsb.edu/~padraic/ucsb_2013_14/mathcs103_s2014/mathcs103_s2014_zhixing_presentation.pdf` (visited on 05/30/2018).

[3] Dave Milovichand and Eric Weisstein. "Nowhere Dense". In: (June 2018). URL: `http://mathworld.wolfram.com/NowhereDense.html` (visited on 06/04/2018).

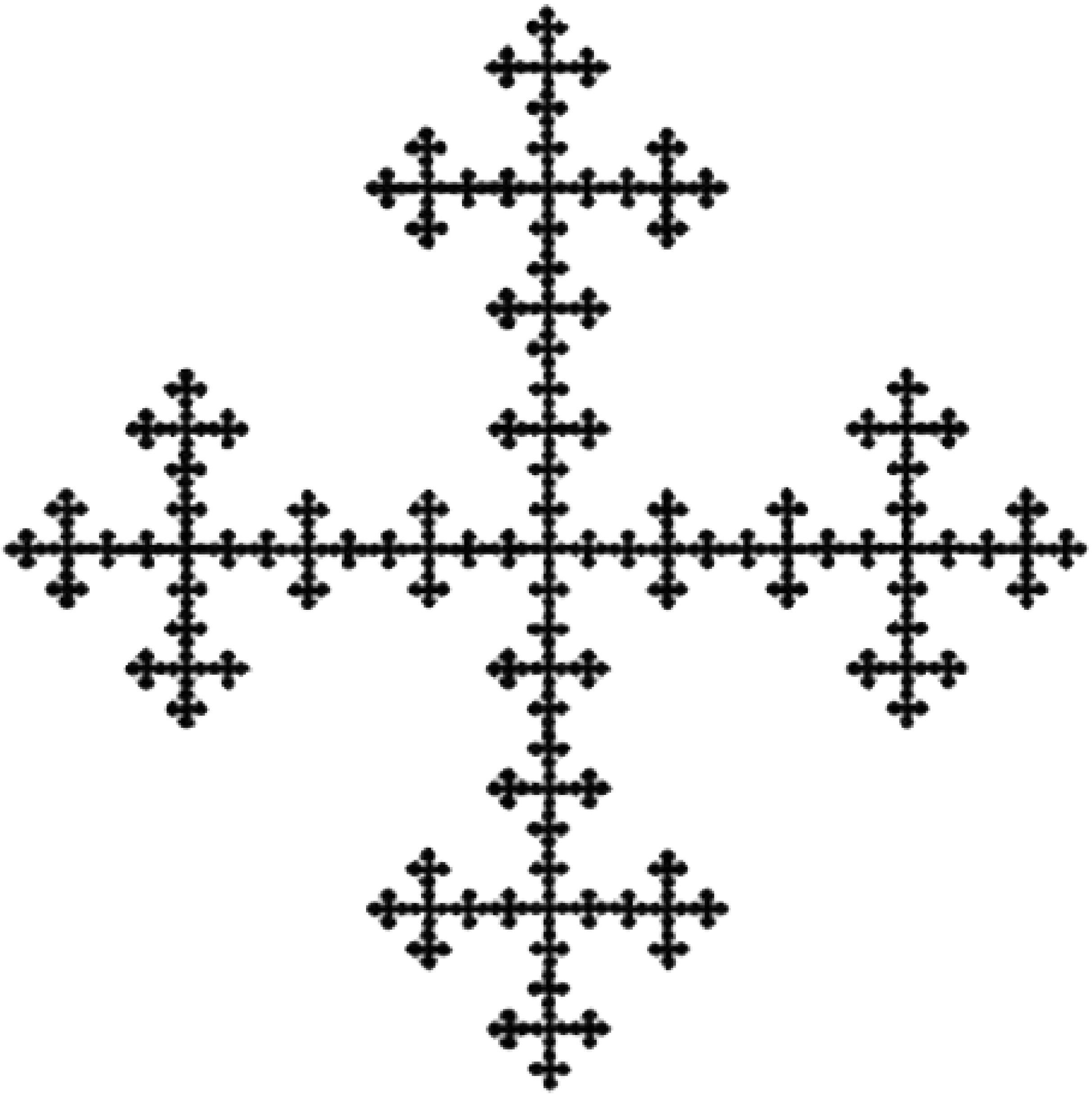

**Figure 2.** A Fractal Model of $\mathbb{Z}_5$

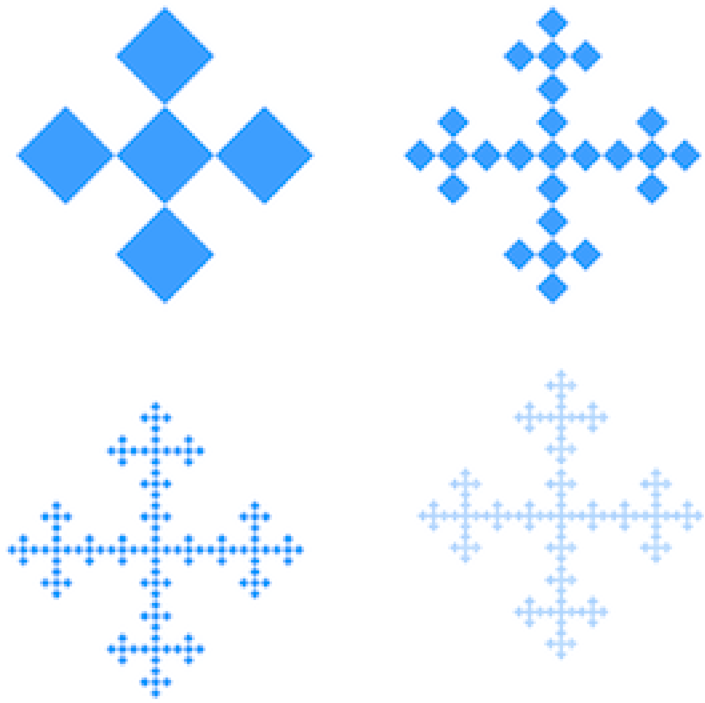

**Figure 3.** Iterations of Figure 2

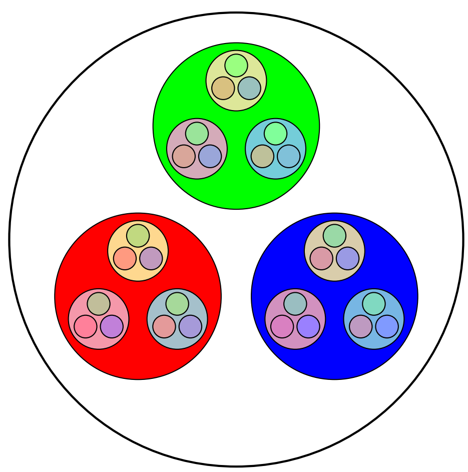

**Figure 4.** The model of $\mathbb{Z}_3$ mentioned in week 2


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[5] Anatole Katok Alexey Sossinsky. "Introduction to Modern Topology and Geometry". In: (Aug. 2007). URL: `http://www.personal.psu.edu/axk29/MASS-11/GT-book-excerpt.pdf` (visited on 06/04/2018).

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