CANTOR SET AND THE P-ADICS

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1. Defining the Cantor Set

To explore the connections between the Cantor Set and the p-adic numbers. It is useful to define and examine the Cantor Set itself.

Definition 1.1. The Cantor Set is iterative constructed as follows. We begin with the segment $[0, 1]$ and split it into thirds. We keep the closed interval $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Then we take our two smaller intervals, split each into thirds and "discard" the open middle interval in each sub interval. We do this infinitely, resulting in 2^n intervals of length 3^{-n}.

There are four main properties of the Cantor Set which are important to observe.

Lemma 1.2.
1. C is perfect. See definition below.
2. C is uncountable
3. C has a vanishing Lebesque Measure
4. C is compact

Definition 1.3. A set X is said to be perfect if $\forall x \in X$ there is a sequence $x_n \in E - \{x\}$ such that x_n converges to x.

Proof. For the first statement we begin with some $x \in C$. We can obviously choose $x_n \subset C - \{x\}$ such that $|x_n - x| \leq 3^{-n}$ where n can be arbitrarily large.

The second statement is implied by the first. Let $E = \{e_i\}_{i=1}^{\infty} \subset \mathbb{R}$ be countable. Also define $E_n = E - \{e_n\}$. Pick $x_1 \in E_1$ and pick a finite open interval I_n such that $x_1 \in I_n$ and $e_1 \notin \overline{I_1}$. Since E is perfect, $I_1 \cap E_2 \neq \emptyset$. Then pick $x_2 \in I_2 \cap E_2$ and let I_2 be an open interval such that $x_2 \in I_2 \subset I_1$ and $e_2 \notin I_2$. Doing this repeatedly we get a sequence of decreasing interval (1_n) such that $e_n \notin I_n$. We also notice that ...

$$\bigcap_{n \geq 1} E \cap I_n \neq \emptyset$$

since all sets $E \cap I_n$ are compact and nonempty. This contradicts our assumption that E is countable.

For the third statement we know that C_n is made up of 2^n segments of length 3^{-n} and therefore $|C_n| = (\frac{2}{3})^n$. Since $C_n \supset C_{n+1}$ for all ...

$$|C| = \lim_{n \xrightarrow{\infty}} |C_n| = \lim_{n \xrightarrow{\infty}} \left(\frac{2}{3}\right)^n = 0.$$

The last statement is proven by the Heine Borel theorem that states that if $C \subseteq \mathbb{R}$ then it is closed and bounded if and only if it is compact. The Cantor set is defined to be closed and is clearly bounded as $\forall x \in C : d(0, x) \leq 1$, and is thus compact. ∎

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2. Variants of the Cantor Set and an Explicit Homeomorphism

We can also define a variant of the Cantor Set and show its relation to $\mathbb{Z}_p$.

Definition 2.1. For any prime number p we say that $C^{(p)}$ is set made by splitting the closed interval $[0,1]$ into $2p-1$ segments and removing every other segment. Then we do this again to the remaining intervals ans so on and so forth infinitely.

We can now define an explicit homeomorphism between the Cantor set and $\mathbb{Z}_p$. We define function $F : \mathbb{Z}_p \to C$ as ...

$$\sum_{n=0}^{\infty} a_n p^n \to \sum_{n=0}^{\infty} (2a_n)(2p-1)^{-(n+1)}$$

Further we can prove that this transformation and its inverse are continuous.

Theorem 2.2. *The function F stated above is a homeomorphism between C and $\mathbb{Z}_p$.*

Proof. Let $x = \sum_{n=0}^{\infty} x_n(p^n)$ and $y = \sum_{n=0}^{\infty} y_n(p^n)$ be elements of $\mathbb{Z}_p$. If ...

$$|x - y| \leq p^{-k}$$

... then the first k digits of the p-adic expansion of are the same. This in turn implies that the first k digits of $F(x)$ and $F(y)$ are the same meanning ...

$$|F(x) - F(y)| \leq (2p-1)^k$$

Clearly F is continuous as close values stay relatively close. Thus we have shown a homeomorphism. ∎