THE BRUHAT-TITS TREE

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1. Lattices

Before discussing the Bruhat-Tits Tree, we have to what makes up the vertices of the tree, which are lattices.

Definition 1.1. A lattice in $\mathbb{R}^2$ is a subset of $\mathbb{R}^2$ that is isomorphic (there is a mapping) to $\mathbb{Z}^2$ and spans $\mathbb{R}^2$ or, equivalently, all linear combinations of the basis vectors and integer coefficients. So, a lattice L with basis v_1, v_2 of $\mathbb{R}^2$ can be defined as

$$L = \{a_1 v_1 + a_2 v_2; a_i \in \mathbb{Z}\}$$

The most trivial lattice is just $\mathbb{Z}^2$.

Furthermore, these lattices can be extended to the p-adics.

Definition 1.2. A lattice in $\mathbb{Q}_p^2$ is a subset of $\mathbb{R}^2$ that is all linear combinations of the basis vectors and integer coefficients. So, a lattice L with basis v_1, v_2 of $\mathbb{Q}_p^2$ can be defined as

$$L = \{a_1 v_1 + a_2 v_2; a_i \in \mathbb{Z}_p\}$$

Again, the most basic lattice is $\mathbb{Z}_p^2$.

2. Bruhat-Tits Tree

The Bruhat-Tits Tree is a graph whose vertices are the equivalence classes of lattices in $\mathbb{Q}_p^2$. We define this equivalence relation $\sim$ as

$$L \sim L' \iff L' = \alpha L \text{ for some } \alpha \in \mathbb{Q}_p$$

In other words, two lattices are equivalent if they are similar. Next, we have to define the edges of the tree. Two equivalence classes have an edge between them if there exist some representatives A, B of each respective equivalence class such that

$$pA \subset B \subset A$$

Note that this also implies that $pB \subset pA \subset B$, which means that this graph is undirected. It turns out we can also describe the types of nodes that are connected to some node. Similar to how $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$, all lattices A have that $A/pA \cong \mathbb{F}_p^2$. Therefore, for lattices B such that $pA \subset B \subset A$, B/pA is isomorphic to a 1 dimensional subspace (a line) of $\mathbb{F}_p^2$, which there are $p+1$ of. These are $(a, 1); a \in \mathbb{F}_p$ and $(0, 1)$. Thus, every node in the graph has exactly $p+1$ edges.

Furthermore, we can describe the neighbours of any node M. Take the basis of some representative of M (u, v), then the neighbors of M are the equivalence classes of the lattices generated by $(pu, xu + v); x \in \mathbb{Z}_p/p\mathbb{Z}_p$ and also (u, pv).

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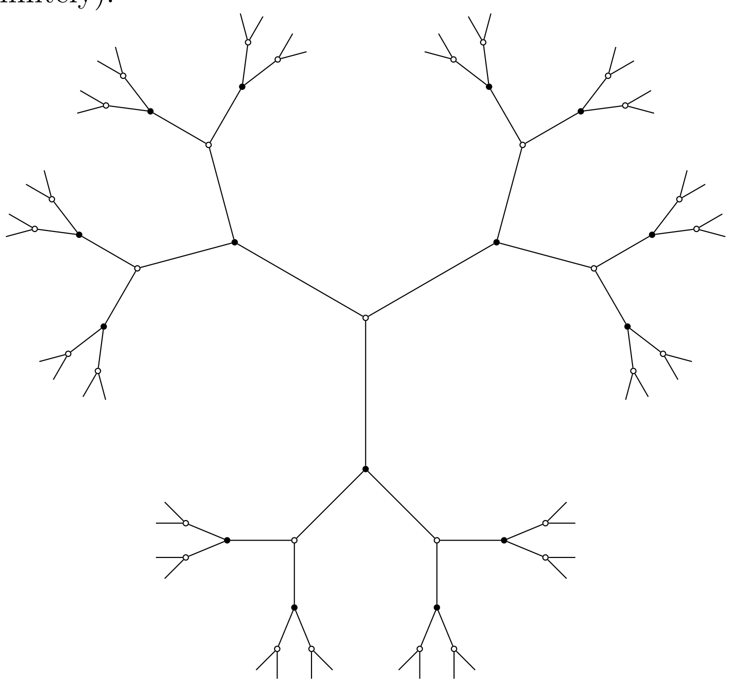
Finally, let's prove that the graph is actually a connected graph (there is a path between every pair of vertices. It is sufficient to prove that the there is a path from the equivalance class of $\mathbb{Z}_p^2$ to any node. Take some lattice M of the destination node. Then we can find a basis (v, g) of $\mathbb{Z}_p^2$ such that some $(p^m v, p^n g)$ is a basis of M. Then, with similarity on M, take $m = 0, n \geq 0$. Therefore, there is a chain of lattices (and thus, equivalence classes or nodes) generated by basis vectors $((1, 0), p^k(0, 1))$.

We have completely described the tree using the properties that we proved. It is a connected graph, it is a tree, and every node has $p+1$ edges. We can root the tree at some node, say the equivalence class $\mathbb{Z}_p^2$ and draw and branch out. $p = 2$ is shown below (continuing infinitely).



3. Matrices

It turns out the vertices and edges of the tree have an analogue with matrices with p-adic entries. We give a very brief overview here.

Definition 3.1. The general linear group of degree n over some field S, denoted $GL_n(S)$, is the set of $n \times n$ invertible matrices with entries in S and matrix multiplication as the group operation.

In particular, the determinant must be non-zero in order to be invertible. Consider the general linear group $GL_2(\mathbb{Q}_p)$ and $GL_2(\mathbb{Z}_p)$. It's not hard to intuitively see the connection between these and lattices, as lattices are generated by bases, which must be independent vectors. Although we won't prove it here, $GL_2(\mathbb{Q}_p)$ acts transitively on both the edges and the vertices of the tree. Furthermore, it turns out the vertices of the Bruhat Tits Tree are in bijection with $GL_2(\mathbb{Q}_p)/GL_2(\mathbb{Z}_p)$.