p-ADIC ANALYSIS: KRASNER'S LEMMA

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1. Krasner's Lemma

Theorem 1.1 (Krasner's Lemma). *Let $f(x) \in \mathbb{Q}_p[x]$ be an irreducible polynomial, and let $\alpha = \alpha_1, ..., \alpha_n$ be all of its roots, in some extension field of $\mathbb{Q}_p$. Suppose that β is a root of some nonzero polynomial in $\mathbb{Q}_p[x]$, and that $|\alpha - \beta| < |\alpha - \alpha_i|$ for $2 \leq i \leq n$. Then $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\beta)$.*

Krasner's lemma roughly says that if you change the coefficients of a polynomial a little, the field of a root will not change. Since the roots of a polynomial continuous functions of the coefficients, if you change the coefficients of $f(x)$ just a little to get $g(x)$, and β is the root of $g(x)$ closest to β, then $|\alpha - \beta|$ will be very small. Since $|\alpha - \alpha_i|$ is constant, it follows that if you change the coefficients by a sufficiently small amount, then you end up with the same field.

2. Applications of Krasner's Lemma

Definition 2.1. A set S of real numbers is *compact* if every sequence in S has a subsequence that converges to an element in S.

Proposition 2.2. *A set S of real numbers is compact if and only if the set is closed and bounded.*

Corollary 2.3. *$\mathbb{Q}_p$ has only finitely many extensions of each degree.*

Proof. Given by Krasner's Lemma, if we take all the monic irreducible degree-n polynomials in $\mathbb{Z}_p[x]$ then for each $f(x)$, there must be some positive number ε_f so that if we change each of the coefficients other than the leading term by less than ε_f in the p-adic sense, then we end up with the same field. We can represent each polynomial $f(x)$ by an n-tuple of elements of $\mathbb{Z}_p$, namely the coefficients. So inside $\mathbb{Z}_p^n$, we have all these polynomials, together with these $\varepsilon_f - balls$ around each one. These $\epsilon_f - balls$ cover all of $\mathbb{Z}_p^n$. We only want the irreducible polynomials, not all polynomials, so this won't actually be all of $\mathbb{Z}_p^n$. This makes a small difference, but the rest of the argument still works with this slightly modified space.

Since the space $\mathbb{Z}_p^n$ is compact, we know that if we cover it by any number of open balls, then we can actually find only finitely many of those open balls that already cover $\mathbb{Z}_p^n$. So, taking our finite number of ε_f'balls that cover all of $\mathbb{Z}_p^n$, each one of those corresponds to a single degree-n extension. Those are all the different degree-n extensions, so there are only finitely many. ∎

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