Grunwald-Wang Theorem

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Abstract

1 Theorem 1

Let K be a number field, $\alpha \in K$, $n \in Z_+$. Then $\alpha \in K_p^n$ for almost every prime p of K if one of the following holds: (i) $\alpha \in K^n$ (ii) $8|n$ and $\alpha \in 2^{\frac{n}{2}}$

2 Corollary 1.1

Let $\alpha \in Q$, $n \in Z_+$. Then α is an nth power almost everywhere locally if one of the following holds: (i) $\alpha \in Q^n$ (ii) $8|n$ and $\alpha \in 2^{\frac{n}{2}} Q^n$

2.1 Proof of why 16 doesn't work

The first element that doesn't hold true is 16. This happens because 16 is an 8th power modulo every prime, since $x^8 - 16 = (x^2 - 2)(x^2 + 2)(x^2 + 2x + 2)(x^2 - 2x + 2)$ and modulo each odd prime is either 2 is a square, -2 is a square, or -1 is a square, in which case the last 2 quadratics factor modulo p. This means that all of these factors of the equation is that each factor square modulo p is -2,2, or -1.

2.2 Lemma 2.1

If $a \in K_p^n$, then $\alpha^{\frac{n}{2}}$m .

2.3 Proof of Lemma 2.1

For all but finitely many primes p of K, p splits completely in $K(\zeta_n, \alpha^{\frac{1}{n}})$ if it splits completely in $K(\zeta_n)$ and $x^n \equiv \alpha \pmod{p}$ is solvable. If p splits completely in $K(\zeta_n, \alpha^{\frac{1}{n}})$; then it also does in $K(\zeta_n)$ and in $K(\alpha^{\frac{1}{n}})$, this implies that $x^n \equiv \alpha \pmod{p}$ is solvable.