# SPECIAL P-ADIC ANALYTIC FUNCTIONS

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**Definition 0.1.** Define $s_p(n) = \sum_i n_i$, where $n = \sum_i n_i p^i$, so basically the sum of the digits. Now, we define a special analytic function in $\mathbb{C}_p$ to be $\sum_i a_i z^i$, with the requirement $v_p(a_n) - \frac{s_p(n)}{p-1} \to \infty$. Basically, the number of primes grows faster than the number of digits in the subscript, roughly.

We can then just let $S(f) = \inf_n\{v_p(a_n) - \frac{s_p(n)}{p-1}\}$. Now, clearly $v_p(f(z)) \geq S(f)$ for all $z \in \mathbb{Z}_p$.

Now, define $SA$ to be the $\mathbb{C}_p$-Banach algebra of all special analytic functions under point wise addition and multiplication and valuation $p^{-S}$. You may ask, what is a $\mathbb{C}_p$-Banach algebra? Well, this is when you have an associative algebra, like $\mathbb{C}_p$, with addition together, and multiplication of scalars, and also a norm/valuation, satisfying the following properties:

$$||x|| \geq 0, ||x|| = 0 \iff x = 0$$

$$||\alpha x|| = |\alpha| ||x||$$

$$||x + y|| \leq ||x|| + ||y||$$

$$||xy|| \leq ||x|| ||y||.$$

First let's prove that this works. For the first one, we know that the exponent function is non-negative, with zero only if $S(f) = \infty$, or $f(z) = 0$. The second one is true because $S(zf) = \inf_n v_p(a_n z) - \frac{s_p(n)}{p-1} = v_p(z)S(f)$, and so $e^{S(zf)} = |z|e^{S(f)} = |z|||f||$. The third one is true because we know that $|a + b| \leq |a| + |b|$, or $v_p(a + b) \geq \min(v_p(a), v_p(b))$, and so each term has larger valuation, and so $S(x + y) \geq \min(S(x), S(y))$, which gets that $||x + y|| \leq \max(||x||, ||y||)$. And then for the last one,

$$S(xy) = \inf_n\{v_p(x_n y_n) - \frac{s_p(n)}{p-1}\} = \inf_n\{v_p(x_n) + v_p(y_n) - \frac{s_p(n)}{p-1}\}$$

$$\geq \inf_n\{v_p(x_n) - \frac{s_p(n)}{p-1} + v_p(y_n) - \frac{s_p(n)}{p-1}\} \geq \inf_n\{v_p(x_n) - \frac{s_p(n)}{p-1}\} + \inf_n\{v_p(y_n) - \frac{s_p(n)}{p-1}\} = S(x) + S(y)$$

, and since $S(xy) \geq S(x) + S(y)$, $e^{-S(xy)} \leq e^{-S(x)-S(y)}$, or $||xy|| \leq ||x||||y||$. Now, we need to show the fact that this addition and multiplication is commutative, association, those things. However, since this is term wise, this works since $\mathbb{C}_p$ works. Henceforth, $SA$ is a Banach algebra, is what I would say given that I know that addition and multiplication is closed and so is scalar multiplication, as $S$ requires less to be defined than a special analytic function. However, since addition and multiplication can only be increasing the valuation compared to the lowest term, and so each term which $S$ minimizes just increases after addition and multiplication. Finally, this is the same with scalar multiplication. So, this is a $\mathbb{C}_p$-Banach algebra.

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My resource also said that this is closed under derivatives and integrals. Now, clearly this is closed under derivatives, as it is just multiplying the coefficients. Now, taking the integral will make it so that we want $v_p(a_n) - v_p(n+1) - \frac{s_p(n+1)}{p-1} \to \infty$. We are given that $v_p(a_n) - \frac{s_p(n)}{p-1} \to \infty$. Since $v_p(n+1)$ is how many zeros there are, and so how many numbers there aren't at the end, we get $s_p(n) - (p-1)v_p(n+1) + 1 = s_p(n+1)$. If we substitute that, we get that we want $v_p(a_n) - v_p(n+1) - \frac{s_p(n)-(p-1)v_p(n+1)+1}{p-1} \to \infty$, which is just $\frac{1}{p-1}$ less then what we were given approaches infinity by the fact that $a$ is a special p-adic analytic function. Since that approaches infinity, and what we wanted is just a constant below that, special p-adic analytic functions are closed under integration. Anyway, he also gave four more identities:

$$S(f+g) \geq \min(S(f), S(g))$$

which is true because this is true for the p-adic addition, and

$$S(f \times g) \geq S(f) + S(g)$$

because again this is true for multiplication, and

$$S(f') \geq S(f) + \frac{1}{p-1}$$

which takes a bit more thought, but is true because of the fact that the integral was $\frac{1}{p-1}$ less, and

$$S(\int f) = S(f) - \frac{1}{p-1}.$$

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