PISOT NUMBERS

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Pisot Numbers, often referred to as Pisot-Vijayaraghavan Numbers, have several unique number theory applications. First discovered in the research of the uniform distribution of sequences, Pisot Numbers have a variety of interesting properties. They have many applications in the exploration of near integers, uniform distribution, and the determination of whether a number is algebraic. Several proofs that build up to these results are omitted from this paper with reference to their location. The Pisot-Vijayaraghavan Problem, which is stated and explored at a basic level here, is a major open problem today.

1. Definitions

Definition 1. An algebraic integer of degree n is a root α of an irreducible monic polynomial $P(x)$ of degree n with integer coefficients, its minimal polynomial.

Definition 2. The conjugates of α are the other roots of $P(x)$.

Definition 3. If $\alpha > 1$ but the other roots lie within the unit circle $|x| < 1$ on the complex plane, α is a Pisot Number.

Example. The golden ratio ϕ is a Pisot Number because it is around 1.618... however the other root to its minimal polynomial $x^2 - x - 1$, $-\phi^{-1} = -0.618\ldots$ where $|-0.618\ldots| < 1$.

Definition 4. $||x||$ denotes the distance between x and the nearest integer. Piecewise, this can be viewed as

$$\begin{cases} x \pmod 1 & \text{if } x \pmod 1 \leq 0.5 \\ 1 - x \pmod 1 & \text{if } x \pmod 1 > 0.5 \end{cases}$$

where the least positive value of x $\pmod 1$ is always taken.

Notation. In this paper, θ and α will be frequently used to denote Pisot Numbers, with specification in each section dictating it more precisely.

Notation. The roots of an irreducible monic polynomial $P(x)$ are $\theta^{(1)}, \ldots, \theta^{(s)}$, and one of these is a Pisot Number, then $\theta^{(1)}$ is that Pisot Number; θ in the absence of a superscript is taken to mean $\theta^{(1)}$.

Next are some basic properties of pisot numbers.

2. Basic Properties

Proposition 5. *(By definition) Every integer greater than 1 is a Pisot Number. Proof: Consider the polynomial $f(x) = x - k$ where the root is the integer $k > 1$. There are no other roots but k is still considered to be a Pisot Number.*

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Proposition 6. *Every rational Pisot Number is an integer greater than 1. Proof: This directly follows from the polynomial $P(x)$ having the requirement of having integer coefficients and being monic.*

Proposition 7. *Powers of Pisot Numbers are also Pisot Numbers. Proof: This follows from the fact that the conjugates of a power of some number are just the powers of the conjugates*

3. Pisot Numbers Modulo 1

Theorem 8. *Let θ be a Pisot Number; the sequence θ^n converges to 0 modulo 1.*

Proof. Let $\lambda = \sup_{j=2,\dots,s} |\theta^{(j)}|$. By Newton's Formulas, we know that the sum of any power of the roots of a polynomial with integer coefficients is a polynomial. In particular, $\theta^n + \sum_{j=2}^{s} \theta^{(j)^n}$ is an integer. Therefore, we know for large values of n that $||\theta^n|| = |\sum_{j=2}^{s} \theta^{(j)^n}|$. We find that $\theta^{(1)^n} + \cdots + \theta^{(s)^n} \leq \lambda^n + \dots \lambda^n = (s-1)\lambda^n$. Therefore, because it is a geometric sequence, the sequence $||\theta^n||$ converges to 0. ☐

4. $\lambda\theta^n$ (Mod 1)

Theorem 9. *Let θ be a Pisot Number and λ be an algebraic integer of $\mathbb{Q}(\theta)$. Then, the sequence $||\lambda\theta^n||$ converges to 0.*

Proof. Let λ^j denote an algebraic integer of $\mathbb{Q}(\theta^j)$. As before, $\lambda\theta^n + \sum_{j=2}^{s} \lambda^{(j)}\theta^{(j)^n}$ is an integer such that $||\lambda\theta^n|| = |\sum_{j=2}^{s} \lambda^{(j)}\theta^{(j)^n}|$. With the same argument as last time, it is clear $||\lambda\theta^n||$ converges to 0 geometrically. ☐

5. The Converse Statement (The Pisot-Vijayaraghavan Problem)

The converse statement to theorem 7 (above) is an open question known as the Pisot-Vijayaraghavan Problem. It asks that if θ is an integer greater than 1, λ is a non-zero real, and $\lim_{n\to\infty} ||\lambda\theta^n|| \to 0$, then is θ a Pisot Number? There are two weaker statements that are known but we will not prove (proofs of both can be found in [Bea92]):

Theorem 10. *If θ is algebraic, then it is a Pisot Number.*

Theorem 11. *If $||\lambda\theta^n||$ converges to 0 sufficiently rapidly, θ is a Pisot Number.*

[Bea92] point out that both imply that $\lambda \in \mathbb{Q}(\theta)$.

6. Small Pisot Numbers

Theorem 12 (Siegel)**.** *The smallest Pisot Number is*

$$\theta_0 = \frac{1}{6}\left(\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}\right) 2^{\frac{2}{3}} \sqrt[3]{3} \approx 1.3247\dots$$

which is the only real root of $x^3 - x + 1$.

This turns out to be quite difficult to prove and requires first proving that the set of Pisot Numbers is closed, which is an important but also complicated insight. A proof of this being the smallest value is found in [Sie44] and the proof that the set of Pisot Numbers is closed was found by [Sal44].

7. Applications of Pisot Numbers

7.1. **Uniform Distribution.** Uniform Distribution was the context in which Pisot Numbers were discovered, and their role has always been important in the development of that theory [Bea92].

Theorem 13. *Let α be a real number greater than 1. Then the following statements are equivalent:*

- *α is not a Pisot Number.*
- *There exists an integer $q > \alpha$ such that the sequence $u_n(\alpha, q)$ is uniformly distributed mod 1.*
- *For all integers $q > \alpha$, the sequence $u_n(\alpha, q)$ is uniformly distributed mod 1*

where $u_n(\alpha, q) = (q - \alpha) \sum_k \{\frac{n}{q^k}\}\alpha^k$ and $\{\frac{a}{b}\}$ indicates the fractional part of $\frac{a}{b}$.

We can prove this given three Lemmas that we will not prove.

Lemma 14 (Weyl's Critereon)**.** *A sequence $x_1, x_2, ...$ is uniformly distributed modulo 1 if and only if*

$$lim_(N \to \infty)\frac{1}{N} \sum_{n<N} e^{2\pi i m x_n} = 0$$

for all m.

A proof of this is found in [Fin03]

Lemma 15. *A real θ greater than 1 is a Pisot Number if and only if there exists a non-zero real λ such that the series $\sum_{n=0} ||\lambda\theta^n||^2$ converges.*

A proof of this is found in [Bea92].

Lemma 16. *Let (a_k) be a sequence of real numbers, and q a positive integer such that the series $\sum_k a_k q^{-k}$ converges. We set $\rho_k = \sum_{h=k+1} a_h q^{-h}$ and $v_n = \sum_k \{\frac{n}{q^k}\} a_k$. Then,*

$$lim\ sup_{x\to\infty} \frac{1}{x} |\sum_{n}^{k} e^{2i\pi v_n}| \leq \prod_k |\frac{sin(\pi q^{k+1}\rho_k)}{q sin(\pi q^k \rho_k)}|.$$

This is proven in [Bea92].

Proof of Theorem 13. Let $a = j(q - \alpha)\alpha^k$ for some positive integer j. The series $\sum_k^n a_k q^{-k}$ converges, and it follows from the third lemma that

$$\limsup_{x\to\infty} \frac{1}{x} |\sum_{n}^{k} e^{2i\pi u_n(\alpha,q)}| \leq \prod_k |\frac{\sin(\pi j q \alpha^{k+1})}{q \sin(\pi j \alpha^{k+1})}|.$$

Suppose α is not a Pisot Number; then by the second lemma, the series $\sum_k ||j\alpha^k||^2$ diverges and consequently, the infinite product on the right hand side does as well. Therefore, by Weyl's Critereon, the sequence is uniformly distributed modulo 1. Suppose α is a Pisot Number; then the infinite product is non-zero and sequence is not uniformly distributed modulo 1. If one of the factors is zero then there exists a k_i such that $j\alpha^{k_i} \in \{\frac{1}{q} \dots \frac{q-1}{q}\}$; then α^{k_0} is rational. Because it is also a Pisot Number, it must then be an integer such that α is also an integer and $u_n = n(q-\alpha)\sum_k \frac{\alpha}{q}^k = \alpha n$ such that $u_n(\alpha, q)$ is not uniformly distributed modulo 1. □

7.2. **Near-Integers.** We know that $||\theta^n|| \to 0$ and $||\lambda\theta^n|| \to 0$, so we can use this to easily generate near-integers. ther

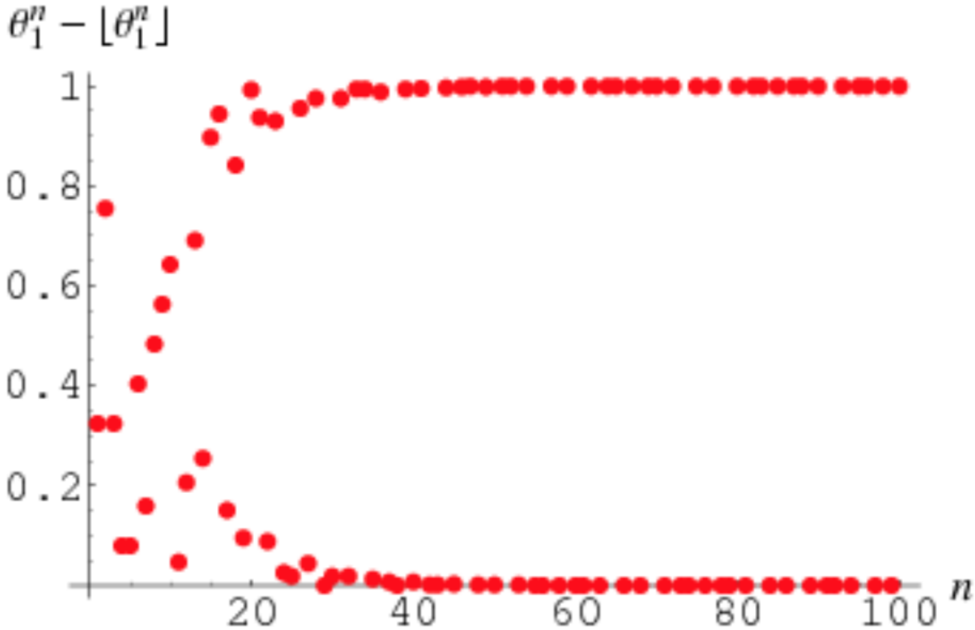

Figure 1. Pisot numbers and near integers

Example. The minimal polynomial $x^2 - 6x - 1$ has roots $3+\sqrt{10}$ and $3-\sqrt{10}$ such that the former is a Pisot Number.

$$(3+\sqrt{10})^6 = 27379 + 8658\sqrt{10} = 54757.9999817\cdots \approx 54758 - \frac{1}{54758}.$$

7.3. **Approximating irrationals.** In a very similar way as the previous application, Pisot Numbers can be used to approximate rationals as the radical term and the integer term will also tend to converge.

Example. In the above example, 27379 is very close to $8658\sqrt{10}$ such that

$$\frac{27379}{8658} = 3.162277662\cdots \approx \sqrt{10} = 3.162277660\ldots.$$

References

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