NUMBER THEORY WEEK 10: CONTINUED FRACTIONS

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1. Introduction

Continued fractions are an interesting way of approximating rational and irrational numbers. Not only do they have some interesting properties that make then useful for representing real numbers, but the manner in which they are created exactly parallels the Euclidean Algorithm. In this paper, we will begin in section two by outlining some basic terms used for communication about continued fractions. For example, we will address exactly what a continued fraction is defined to be. In section three, we will discuss the Euclidean Algorithm and one aspect of its relationship with continued fractions. In the fourth section, we will prove a couple of theorems about representing real numbers, rational and irrational, as continued fractions. Namely, we will prove that a number is rational if and only if it can be expressed as a simple continued fraction and that any irrational number is equal to the limit of an infinite continued fraction (terms to be defined in section two). Finally, we will conclude with some interesting examples of irrational, namely transcendental, numbers and their continued fraction expansions.

2. Definitions and Notation

First of all, we need to define a continued fraction.

Definition 2.1. A *continued fraction* is of the form

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots}}$$

[4]

Definition 2.2. A continued fraction of the above form where $b_i = 1$, $a_0 \in \mathbb{Z}$, and a_i is a positive integer for all positive integers i is called a *simple continued fraction.* [4]

We will only be studying simple continued fractions in this paper.

Definition 2.3. A continued fraction where there are finitely many a_i, b_i is called a *finite continued fraction.* Alternately, a continued fraction that never terminates is called an *infinite continued fraction.*[4]

For notation purposes, we will denote

$$[a_0; a_1, a_2, a_3, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Definition 2.4. If we have the continued fraction $A = [a_n; a_{n-1}, a_{n-2}, a_{n-3}, \dots a_0]$, and we label our term such that $\frac{1}{a_0} = c_0$, $\frac{1}{a_1 + c_0} = c_1$, etc., c_k is called the *kth value* of A.

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Definition 2.5. The continued fraction $[a_0; a_1, a_2, \ldots, a_k]$ where k is a non-negative integer less than or equal to n is called the *kth convergent* of the continued fraction $[a_1; a_2, a_3, ..., a_n]$. The kth convergent is denoted by C_k. [4]

Note that a_0 is the only term outside of the fraction. If $a_0 = 0$, the value of the entire continued fraction will be less than one but greater than zero. If $a_0 \geq 1$, the continued faction will be greater than one. It is trivial to prove this, but we include the proof anyway.

Proof. We have the continued fraction $A = [a_n; a_{n-1}, a_{n-1}, a_{n-3}, ...a_0]$. Let us label our term such that $\frac{1}{a_0} = v_0$, $\frac{1}{a_1+v_0} = v_1$, etc, such that the v_i's are the ith values of A. We can prove our statement using induction. Since a_0 is an integer greater than or equal to 1, v_0 will be greater than zero and less than or equal to one. We will assume that for some $i > 0$, v_i is greater than zero and less than or equal to one. Thus, since a_i is also an integer greater than or equal to 1, $a_{i+1} + v_i$ will be greater than one. Thus, this sum's reciprocal, v_{i+1}, will be less than one and greater than zero. Thus, by induction, the final $v_n = A - a_n$ will be less than one but greater than zero. Ergo, if $a_n \geq 1$, $A \geq 1$, and if $a_n = 0$ (it cannot be between zero and one, since it must be a non-negative integer), $0 < A < 1$. ■

3. The Euclidean Algorithm [2]

It's worth noting that the Euclidean algorithm is related to continued fractions. To see this, consider integers a, b where $a, b \in \mathbb{Z}$ and $b \neq 0$. We have

$$\frac{a}{b} = n_1 + u_1, u_1 = \frac{a - n_1 b}{b}$$

where $0 \leq u_1 < 1$. So we can see that $u_1 = \frac{r_1}{b}$ where r_1 is the remainder when dividing b into a. Doing something similar,

$$\frac{b}{r_1} = n_2 + u_2, u_2 = \frac{b - n_2 r_1}{r_1} = \frac{r_2}{r_1}$$

where r_1 is the remainder when dividing r_1 into b. So, the successive quotients in the Euclidean algorithm are in fact $n_1, n_2, \ldots$. Since the Euclidean algorithm terminates in a zero after a finite number of steps, the continued fraction expansion of any rational must be finite.

4. Real Numbers as Continued Fractions

We can start by noting that any real number can be written as a continued fraction. If the number is rational, its continued fraction representations are finite [4]. If the number is irrational, its representation is infinite [3].

Example. Consider the real number $\pi = 3.1415926\ldots$. Let $a_0 = 3$, giving us a remainder of $0.1415926\ldots$. So

$$\pi = 3 + 0.1415926\ldots.$$

Further, we can put the second term in continued fraction form to make

$$\pi = 3 + \frac{1}{7.06251\ldots}.$$

We can continue the process to get

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \ldots}}}$$

It's worth noting that the continued fraction ends when the remainder is 1, such as in the following example.

Example. $\frac{56}{13} = 1 + \frac{1}{3+\frac{1}{4}}$

Every rational number can be represented as a finite continued fraction in exactly two different ways.[2] The only difference between the two ways is that the last a_i, call it a_k, is changed to $(a_k - 1) + \frac{1}{1}$. Additionally, every finite continued fraction represents a rational number. That brings us to a theorem.

Theorem 4.1. *A number is rational if and only if it can expressed as a simple finite continued fraction. [4]*

Proof. [4] Let $n = p/q$ be a rational number expressed in lowest terms. Let us apply the Euclidean Algorithm to p and q. This produces:

$$p = a_1 q + r_1, 0 \leq r_1 < q$$

$$q = a_2 r_1 + r_2, 0 \leq r_2 < r_1$$

$$r_1 = a_3 r_2 + r_3, 0 \leq r_3 < r_2$$

$$...$$

$$r_{n-3} = a_{n-1} r_{n-2} + r_{n-1}, 0 \leq r_{n-1} < r_{n-2}, r_{n-2} = a_n * r_{n-1}$$

The sequence $r_1, r_2, r_3, ..., r_{n-1}$ forms a strictly decreasing sequence of non-negative integers that must converge to zero in a finite number of steps. So, there are at most n a_is.
Now, we can rearrange the algorithm into a continued fraction in the following manner.

$$\frac{p}{1} = a_1 + \frac{1}{\frac{q}{r_1}}$$

$$\frac{q}{r_1} = a_2 + \frac{1}{\frac{r_1}{r_2}}$$

$$...$$

$$\frac{r_{n-2}}{r_{n-1}} = a_{n-1} + \frac{1}{\frac{r_{n-1}}{r_n}}, \frac{r_{n-1}}{r_n} = a_n$$

So we can substitute to get

$$n = \frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + ... + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}$$

For the converse, we can prove by induction that if a continued fraction has n terms, then it must be rational. Say we have a finite continued fraction $A = [a_n; a_{n-1}, ...a_0]$. The zeroth value of A is simply $\frac{1}{a_0}$, which is one over an integer and therefore rational. Assume that, for $k < n$, the the kth value, represented by b, is rational. We can rewrite the $k+1$th value as

$$A = \frac{1}{(k+1) + b}$$

An integer plus a rational must result in another rational number, and one over a rational number is another rational number, so the $k+1$th value must also be rational. Therefore, by induction, a finite continued fraction with n terms must be equal to a rational number. ∎

Every irrational number can be represented as an infinite continued fraction. This can be seen by applying the same algorithm for finding finite continued fractions to irrational numbers. It can also be proven.

First, we will need a lemma.

Lemma 4.2. *[3] $q_k \geq k$ for all $k \geq 1$*

Proof. We will induct on k. $q_1 = a_1 \geq 1$, so the result holds for the base case $k = 1$. Take $k > 1$, and assume it holds for numbers $\leq k$. We will prove that it holds for $k + 1$.

$$q_{k+1} = a_{k+1}q_k + q_{k-1} \geq a_{k+1} \cdot k + (k-1) \geq 1 \cdot k + (k-1) = 2k - 1 = k + (k-1) \geq k + 1.$$

The last inequality used $k \geq 2$. Thus, by induction, our lemma is true. [3] ■

Theorem 4.3. *Let $x \in \mathbb{R}$ be irrational. Let $x_0 = x$, and*

$$a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \text{ for } k \geq 0.$$

Then

$$x = [a_0; a_1, a_2, \ldots].$$

[3]

Proof. First, we will prove by induction that x_k is irrational for all $k > 0$.

Since x is irrational and $x_0 = x$, x_k is irrational for $k = 0$. Assume that $k > 0$ and that x_k is irrational for $k - 1$. We will prove that x_k is irrational. Suppose, on the contrary, that $x_k = \frac{s}{t}$, where $s, t \in \mathbb{Z}$. Then

$$\frac{s}{t} = \frac{1}{x_{k-1} - a_{k-1}} \quad \text{so} \quad x_{k-1} = a_{k-1} + \frac{t}{s}.$$

a_k is always an integer since $a_k = [x_k]$, which always outputs integers. So, $a_{k-1} + \frac{t}{s}$ is the sum of an integer and a rational number. Therefore, it is rational, so x_{k-1} is rational, producing a contradiction. Thus, if x_{k-1} is irrational, x_k is irrational. By induction, x_k is irrational for all $k \geq 0$.

Next, we will prove that a_k is are positive integers for $k \geq 1$. We already know that the a_k 's are integers.

Let $k \geq 0$. Since $a_k = [x_k]$, the definition of the ceiling function gives

$$a_k \leq x_k < a_k + 1.$$

But x_k is irrational, so $a_k \neq x_k$. Hence,

$$a_k < x_k < a_k + 1,$$

$$0 < x_k - a_k < 1,$$

$$x_{k+1} = \frac{1}{x_k - a_k} > 1,$$

$$a_{k+1} = [x_{k+1}] \geq 1.$$

Since $k \geq 0$, this proves that the a_k 's are positive integers for $k \geq 1$. Now, we will prove that

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} [a_0; a_1, \ldots, a_k] = x.$$

First, we will create a formula for x in terms of the p's, q's, and a's.

Then we will find $\left| x - \dfrac{p_k}{q_k} \right|$ and show that it is less than something which goes to 0.

To obtain the formula for x, start with

$$x_{k+1} = \frac{1}{x_k - a_k}.$$

Rearranging, we get

$$x_k = a_k + \frac{1}{x_{k+1}}.$$

Write out this equation for a few values of k:

$$x_0 = a_0 + \frac{1}{x_1} x_1 = a_1 + \frac{1}{x_2} x_2 = a_2 + \frac{1}{x_3}$$

Substituting the second equation of the set into the first gives

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}}.$$

Substituting $x_2 = a_2 + \dfrac{1}{x_3}$ into this equation gives

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}}.$$

In general,

$$x = x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_k + \cfrac{1}{x_{k+1}}}}}$$

In other words, the x_k 's are the "infinite tails" of the continued fraction.

Recall the recursion formulas for convergents:

$$p_k = a_k p_{k-1} + p_{k-2} \quad \text{and} \quad q_k = a_k q_{k-1} + q_{k-2}.$$

The right sides only involve terms up to a_k and p's and q's of smaller indices. Therefore, the fractions

$$[a_0; a_1, a_2, \ldots, a_k, x_{k+1}] \quad \text{and} \quad [a_0; a_1, a_2, \ldots, a_k, a_{k+1}, \ldots]$$

have the same p's and q's through index k.
Using the recursion formula for convergents, we obtain

$$x = x_0 = [a_0; a_1, a_2, \ldots, a_k, x_{k+1}] = \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}}.$$

Therefore,

$$x - \frac{p_k}{q_k} = \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}} - \frac{p_k}{q_k} = \frac{x_{k+1}p_kq_k + p_{k-1}q_k - x_{k+1}p_kq_k - p_kq_{k-1}}{(x_{k+1}q_k + q_{k-1})q_k} =$$

$$\frac{p_{k-1}q_k - p_kq_{k-1}}{(x_{k+1}q_k + q_{k-1})q_k} = \frac{(-1)^k}{(x_{k+1}q_k + q_{k-1})q_k}.$$

Take absolute values:

$$\left|x - \frac{p_k}{q_k}\right| = \frac{1}{(x_{k+1}q_k + q_{k-1})q_k}.$$

Now

$$x_{k+1} > [x_{k+1}] = a_{k+1}, \quad \text{so} \quad x_{k+1}q_k + q_{k-1} > a_{k+1}q_k + q_{k-1} = q_{k+1}.$$

Therefore,

$$\frac{1}{x_{k+1}q_k + q_{k-1}} < \frac{1}{q_{k+1}},$$

$$\frac{1}{(x_{k+1}q_k + q_{k-1})q_k} < \frac{1}{q_{k+1}q_k},$$

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_{k+1}q_k}.$$

By our lemma, $q_k \geq k$ and $q_{k+1} \geq k+1$, so

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_{k+1}q_k} \leq \frac{1}{k(k+1)}.$$

Now $\lim_{k\to\infty} \frac{1}{k(k+1)} = 0$, so, by the squeeze theorem,

$$\lim_{k\to\infty} \left|x - \frac{p_k}{q_k}\right| = 0.$$

This implies that

$$\lim_{k\to\infty} \frac{p_k}{q_k} = x.$$

[3] ■

We can also prove that an infinite continued fraction must be irrational just by using Theorem 2.1; we know every real number r has a continued fraction representation, and if the representation is finite r must be rational, so if it is not, then its representation must be infinite.

5. Interesting Continued Fraction Representations

A classic example of an infinite continued fraction is the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\dots}}} = [1; 1, 1, 1, 1 \dots]$$

We can prove this by setting the infinite continued fraction equal to x, and then recognizing that the portion under the first fraction bar is the exact same infinite continued fraction. Thus,

$$x = 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\dots}}} = 1 + \frac{1}{x}$$

$$x = 1 + \frac{1}{x}$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

When we solve the quadratic equation for this and discard the negative solution, we arrive at the golden ratio.

Another interesting continued fraction is the (periodic!) continued fraction expansion for e:

$$e = 1 + \cfrac{1}{0 + \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{4\dots}}}}}}} = [1; 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, ...]$$

A short, but difficult, proof can be found at [1]

References

[1] Henry Cohn. "A short proof of the simple continued fraction expansion of e". In: *The American Mathematical Monthly* 113.1 (2006), pp. 57–62.

[2] William F Hammond. "Continued Fractions and the Euclidean Algorithm". In: *University at Albany, Albany, Lecture* (1997). URL: `https://www.math.u-bordeaux.fr/~pjaming/M1/exposes/MA2.pdf`.

[3] Bruce Ikenaga. *Infinite Continued Fractions*. 2010. URL: `http://sites.millersville.edu/bikenaga/number-theory/infinite-continued-fractions/infinite-continued-fractions.html`.

[4] Adam Van Tuyl. "Introduction to Continued Fractions". In: (Jan. 1996). URL: `http://archives.math.utk.edu/articles/atuyl/confrac/intro.html`.