THE STERN-BROCOT TREE

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1. Introduction

The Stern-Brocot tree is a way of organizing the positive rational numbers. It turns out to enumerate every single positive rational exactly once, and it is also a binary search tree. The goal of this paper is to provide an elementary understanding and background of the Stern-Brocot tree while summarizing and proving the main results. The root of the tree starts at 1, and the parent-child relationship is described in Sections 2 and 3. A summary of the major properties is done in Section 4.

The tree was discovered by Moritz Stern and Achille Brocot in the 19th century. They worked independently of each other. Brocot first used the tree to design gears with a ratio of smooth numbers close to some real number.

2. Continued Fractions

Before discussing the Stern-Brocot tree, we have to first define what continued fractions are.

Definition 2.1. The *continued fraction* form of a positive rational number q is:

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}}$$

$a_0, a_1, a_2, \cdots$ are called the terms or the coefficients of the continued fraction, where a_0 is a non-negative integer and $a_1, \cdots, a_k$ are positive integers. In order to make these representations unique, we have the restriction that $a_k \geq 2$. We write the continued fraction as $q = [a_0; a_1, a_2, \ldots, a_k]$.

In the Stern-Brocot tree, we define the parent of some rational $q = [a_0; a_1, a_2, \ldots, a_k]$ as $[a_0; a_1, a_2, \ldots, a_{k-1}+1]$ if $a_k = 2$ and $[a_0; a_1, a_2, \ldots, a_k - 1]$ otherwise. Conversely, the children of some $q = [a_0; a_1, a_2, \ldots, a_k]$ are $[a_0; a_1, a_2, \ldots, a_k + 1]$ and $[a_0; a_1, a_2, \ldots, a_k - 1, 2]$. The smaller of the two children is the left child, and the larger of the two is the right child.

Example. Starting from the root of the tree, $[1;]$ has children $\frac{1}{2} = [0; 2]$ and $\frac{2}{1} = [2;]$. Continuing, the children of those respectively are $\frac{1}{3} = [0; 3]$, $\frac{2}{3} = [0; 1, 2]$ and $\frac{3}{2} = [1; 2]$, $\frac{3}{1} = [3;]$.

3. Mediants

Definition 3.1. The *mediant* of two rational numbers in lowest terms $\frac{a}{c}, \frac{b}{d}$ is

$$\frac{a+b}{c+d}.$$

We can also define the Stern-Brocot tree using mediants. We define the left child of some rational q as the mediant of q and its closest ancestor smaller than q. If there is no such ancestor, then use $\frac{0}{1}$ as the ancestor. Similarly, we define the right child of some rational q as the mediant of q and its closest ancestor larger than q. If there is no such ancestor, then use $\frac{1}{0}$ (which we think as infinity here) as the ancestor. We start with the root $\frac{1}{1}$ and proceed down the tree.The two definitions of the Stern-Brocot tree are in fact equivalent.

Example. The children of $\frac{1}{1}$ are respectively $\frac{1+0}{1+1} = \frac{1}{2}, \frac{1+1}{1+0} = \frac{2}{1}$. Their children are $\frac{1+0}{2+1} = \frac{1}{3}, \frac{1+1}{2+1} = \frac{2}{3}$ and $\frac{2+1}{1+1} = \frac{3}{2}, \frac{2+1}{1+0} = \frac{3}{1}$.

Mediants have the nice property that they are always in between its two ancestors.

Theorem 3.2. *If $\frac{a}{c} < \frac{b}{d}$, then $\frac{a}{c} < \frac{a+b}{c+d} < \frac{b}{d}$.*

Proof. $\frac{a}{c} < \frac{a+b}{c+d}$ follows from the fact that

$$\frac{a+b}{c+d} - \frac{a}{c} = \frac{bc-ad}{c(c+d)} = \frac{d}{c+d}(\frac{b}{d} - \frac{a}{c}) > 0$$

. Similarly, $\frac{a+c}{b+d} < \frac{b}{d}$ follows from the fact that

$$\frac{b}{d} - \frac{a+b}{c+d} = \frac{bc-ad}{d(c+d)} = \frac{c}{c+d}(\frac{b}{d} - \frac{a}{c}) > 0$$

■

Furthermore, mediants also surprsingly have a fairly small denominator.

Theorem 3.3. *If $\frac{a}{c}, \frac{b}{d}$ satisfy $bc - ad = 1$, then the mediant is the fraction with smallest denominator in the interval $(\frac{a}{c}, \frac{b}{d})$.*

Proof. Take some fraction $\frac{x}{y}$ that is in the interval $(\frac{a}{c}, \frac{b}{d})$. Since $bc - ad = 1$, then there exists integers k_1, k_2 such that $x = k_1 a + k_2 b$ and $y = k_1 c + k_2 d$. If we can prove that k_1 and k_2 are positive, we would be done. This is because it would imply that

$$y = k_1 c + k_2 d \geq c + d.$$

To see why k_1 must be positive,

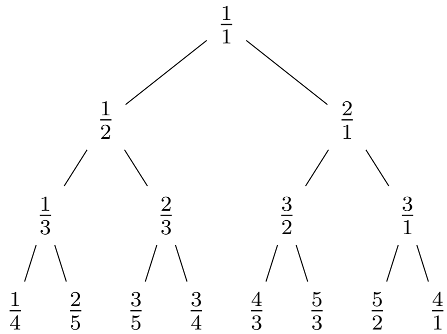
$$\frac{b}{d} - \frac{k_1 a + k_2 b}{k_1 c + k_2 d} = k_1 \frac{bc - ad}{d(k_1 c + k_2 d)}$$

must be positive, which implies that k_1 is also positive. Similarly for k_2,

$$\frac{k_1 a + k_2 b}{k_1 c + k_2 d} - \frac{a}{c} = k_2 \frac{bc - ad}{c(k_1 c + k_2 d)}$$

must be positive, which implies that k_2 is also positive. ■

4. Properties of the Stern-Brocot tree



The Stern-Brocot tree has a number of intersting properties relating to the rationals, the first of which is the following:

Theorem 4.1. *The Stern-Brocot tree contains every positive rational number in lowest terms exactly once.*

Just from the properties of the mediants, we already know that a fraction cannot appear twice in the tree because the construction of the tree using mediants preserves the order of the rationals. To prove the rest of the theorem, we need a couple of lemmas.

Lemma 4.2. *If $\frac{a}{c} < \frac{b}{d}$ and one of the fractions is a parent of the other, then $bc - ad = 1$.*

Proof. Proceed by induction on the distance from the root. The first pairs of fractions $\frac{0}{1}, \frac{1}{1}$ and $\frac{1}{1}, \frac{1}{0}$ work. Assume it holds for some fractions $\frac{a}{c} < \frac{b}{d}$. Then their mediant $\frac{a+b}{c+d}$ also satisfies it for both fractions by some simple algebraic manipulations.

$$c(a+b) - a(c+d) = ca + bc - ac - ad = bc - ad = 1$$

and

$$b(c+d) - d(a+b) = bc + bd - da - db = bc - ad = 1.$$

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Proof of Theorem 4.1. Take some positive rational $\frac{x}{y}$. We need to show that it will appear on the tree. We can find some rationals $\frac{a}{c}, \frac{b}{d}$ that are on the tree and satisfy the above lemma such that $\frac{a}{c} < \frac{x}{y} < \frac{b}{d}$. Sepearating the inequalities,

$$xc - ya > 0 \text{ and } by - dx > 0.$$

Since the variables are all integers, the above is equivalent to

$$xc - ya \geq 1 \text{ and } by - dx \geq 1.$$

Multiplying,

$$(b+d)(xc - ya) \geq (b+d) \text{ and } (a+c)(by - dx) \geq (a+c).$$

Adding the inequalities,

$$(b+d)(xc - ya) + (a+c)(by - dx) \geq a + b + c + d.$$

$$bcx - ady - aby + cdx + aby - cdx + bcy - adx \geq a + b + c + d$$

$$(x+y)(bc - ad) \geq a + b + c + d$$

Applying our lemma,

$$x + y \geq a + b + c + d$$

which means that most after $x + y$ levels of computing mediants, we will reach $\frac{x}{y}$, otherwise, the denominators would get too large. ∎

Definition 4.3. A *binary search tree* is a tree that has a designated root node, where each node stores a value and has a left sub-tree and a right sub-tree. Furthermore, each value in a node must be greater than or equal to any value in the left sub-tree, and less than or equal to any value in the right sub-tree. This allows for a binary search algorithm to be able to correctly traverse the tree.

From Theorem 3.2 and our method of construction, the following becomes clear.

Proposition 4.4. *The Stern-Brocot tree is an infinite binary search tree with respect to the usual ordering of rationals.*

As a result of this, we can find the path from the root to some rational q by using binary search.

The algorithm is as follows: Let $L = \frac{0}{1}$ and $H = \frac{1}{0}$. Until q is either L or H, repeat the following. Compute the mediant M of L and H. If $M < q$, then let $L = M$. Otherwise, let $H = M$.

The values of M are exactly the path from the root to q. We can also approximate decimals with rationals by using this binary search algorithm to arbitrary precision by stopping the search when we have reached a desired precision. By Theorem 3.3, these are the best rational approximation in the sense that their denominators are smallest possible. These approximations are also given by truncating a continued fraction of some real number.