PELL EQUATIONS

1. Introduction

Diophantine equations have been of interest to mathematicians, spanning from Ancient Greece, to India, and now in modern mathematics. One specific type of Diophantine equation that has been studied extensively is the Pell Equation

Pell equations are any Diophantine equation of the form

$$x^2 - ny^2 = 1$$

where n is a squarefree integer. The name itself is sort of a misnomer because Pell never actually discovered the equation, it was attributed to Pell by Euler.

2. Initial Observations

The first thought is that $(\pm 1, 0)$ is a solution. We'll define this as a **trivial** solution. Given that this is a conic curve, once we have this one solutions, there are infinitely many solutions $(x, y) \in \mathbb{Q}^2$. However, the rational solutions aren't that interesting, so when we talk about "solutions" in the rest of this paper, we mean $(x, y) \in \mathbb{Z}^2$.

We obviously want to try and find nontrivial solutions. So, as always, when presented with a Diophantine equation, we try to find solutions for a small n. In this case, we'll look at $n = 2$.

Either using a calculator, or if you're feeling particularly bashy, by hand, we find that some small solutions are $x = 17$ $y = 12$ and $x = 577$ $y = 408$. Note that $\frac{x}{y}$ in both cases is pretty close to $\sqrt{2}$.

As we're computing these solutions, we become sad though, because even though it's relatively easy to compute the first solution and difficult to compute the second, it's basically impossible to find any higher solutions to the equation without the help of a computer. Given this, we want to find a way to generate the solutions to the Pell Equations, which will motivate much of the rest of the paper.

You might be asking why we only consider when n is squarefree, and that is because the if n is a square, then the equation can be rewritten to $x^2 - (d'y)^2 = 1$ and the only sqaures that differ only by one are 0 and 1 giving us the trivial solutions.

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3. Pell Equations Solutions

The first obvious thing we want to figure out is how many solutions such and equation has.

Before we can find the solution, we need to develop some ideas from what we know about continued fractions. A continued fraction, essentially, is something of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2}...}$$

This fraction can either be finite, which denotes a rational number, or infinite, which denotes an irrational number. Because this notation is rather bulky, we will shorten this by assigning something of the above form to $[a_0; a_1, a_2, a_3...]$. I won't go into the proof in this paper, but we know that there exists an infinite continued fraction representation for $\sqrt{n}$ and that it is unique. The reason why this notation is helpful is that if we truncate the continued fraction expansion, we get a rational approximation of the irrational number. These rational numbers are called **convergents**.

Lemma 1.1 There is at least one nontrivial solution solution to any Pell equation.

We'll let $\frac{a_i}{b_i}$ denote the sequence of convergents of the infinite continued fraction expansion of $\sqrt{n}$. Then, if we want to find a solution to Pell's Equation, we just need to find an $x = a_i$ and $y = b_i$ for some i! Furthermore, if we want to find the fundamental solution of a Pell Equation, we can just find convergent where x is the smallest.

Theorem 1.2 There are infinitely many solutions to the Pell equations.

Proof: Essentially we use the observation that

$$(x^2 + dy^2)^2 - d(2xy)^2 = (x^2 - dy^2)^2$$

Because this is true, we plug in the (x, y) we found and the new solution is (a, b) with $a = x^2 + dy^2$ and $b = 2xy$. This allows us to generate infinitely many solutions given a nontrivial solution.

Example 1.1: Use $x^2 - 7y^2 = 1$ and find the first solution, and derive the family of solutions from there.

We can easily calculate by hand that $(8, 3)$ is a solution to the above equation. Using the above formula, we can generate $(127, 48)$ and $(32257, 12192)$. For the first one, we just get $8^2 + 7 \cdot 3^2 = 127$ and $2 \cdot 8 \cdot 3 = 48$. For the second one, $127^2 + 7 \cdot 48^2 = 32257$ and $7 \cdot 127 \cdot 48 = 12192$.

However we can think about the infinite solutions to the Pell Equation differently because it also characterizes all solutions to the Pell Equation nicely.

If we factor the Pell equation, we get

$$(x - y\sqrt{d})(x + y\sqrt{d})$$

This means that finding a solution to the Pell Equation is basically finding a nontrivial unit of the ring $\mathbb{Z}[\sqrt{d}]$ with norm 1. This implies that once you know one solution to Pell's Equation, you know infinitely many of them. This means that if you order all the solutions to the Pell Equations with the n-th solution named as (x_n, y_n), then we can say that the n-th solution to the Pell Equation is just solving for

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

4. Interesting Properties

The solutions to the Pell equations have some interesting properties.

Theorem 2.1 The solutions to the Pell equations approximate the square root of n using the form $\frac{x}{y}$

Proof: Essentially, we'll factor the Pell Equation.

$$x^2 - ny^2 = 1$$
$$\left(\frac{x}{y}\right)^2 - n = \frac{1}{y^2}$$

(4.1)

Factoring the right hand side, we get

$$\left(\left(\frac{x}{y}\right) - \sqrt{n}\right)\left(\left(\frac{x}{y}\right) + \sqrt{n}\right) = \frac{1}{y^2}$$
$$\left(\left(\frac{x}{y}\right) - \sqrt{n}\right)\left(\left(\frac{x}{y}\right) + \sqrt{n}\right) = \frac{1}{y^2}$$
$$\left(\left(\frac{x}{y}\right) - \sqrt{n}\right) = \frac{1}{(y^2)\left(\left(\frac{x}{y}\right) + \sqrt{n}\right)}$$

(4.2)

$$\lim_{y\to\infty} \frac{1}{(y^2)\left(\left(\frac{x}{y}\right) + \sqrt{n}\right)} = 0$$

Because of this, the left hand side also becomes very small. So,

$$\frac{x}{y} = \sqrt{n}$$

Essentially, this shows us that if we generate larger and larger solutions to the Pell Equation, the number $\frac{x}{y}$ starts to approximate $\sqrt{n}$. This is pretty nice, because we can generate large solutions to the Pell Equation quite fast given the mechanism outlined in the last section.

5. Pell-Like Equations

It's pretty obvious that the Pell equations are a rather specific type of Diophantine equation, and there are some obvious generalizations.

- The negative Pell equations, which are the solutions to $x^2 - ny^2 = -1$
- The more general equations $ax^2 \pm bx^2 = c$

There are ways to figure out solutions to the above but are beyond the scope of this paper. They use similar techniques though.