ON FAREY SEQUENCES AND THEIR APPLICATIONS

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Abstract. In this paper, we will explore Farey sequences and some of their key properties. We will also explore some of their most important applications, specifically to rational approximations and Ford Circles. It turns out that irrational numbers can be approximated very well using Farey sequences, and Ford Circles are tangent if and only if their corresponding fractions are adjacent in a Farey sequence.

1. Definition and Key Properties

We first give the definition of the Farey sequence as shown below:

Definition 1.1. We define the *nth Farey Sequence* to be the sequence of nonnegative fractions less than or equal to 1 in increasing order with denominator less than or equal to n when reduced. We denote the nth Farey Sequence as F_n.

Example. For instance the 6th Farey Sequence is

$$F_6 = \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1} \right\}.$$

Now that we have defined the sequence, we can get started with theorems related to these special fractions. Here is our first theorem:

Theorem 1.2. *The size of the Farey sequence F_n is $|F_n| = 1 + \sum_{k=1}^{n} \phi(k)$, where $\phi(k)$ denotes Euler's Totient Function.*

Proof. We will use induction on n, with the base case being obvious. It thus suffices to show that $|F_n| - |F_{n-1}| = \phi(n)$. In other words, we want to show that the number of fractions we add to F_{n-1} to get F_n is $\phi(n)$. Note that we only need to add fractions of the form $\frac{x}{n}$, where $1 \le x \le n-1$. But if $\gcd(x, n) \ne 1$, the fraction can be reduced and is thus already in the sequence. Therefore, we only need to count those x such that $\gcd(x, n) = 1$, and there are $\phi(n)$ of those x, as desired. ∎

The next theorem is very important when we perform computations with Farey sequences:

Theorem 1.3. *If $\gcd(a, b) = \gcd(c, d) = 1$ and $\frac{a}{b} < \frac{c}{d}$, then the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent in some Farey sequence if and only if $bc - ad = 1$.*

We will present a proof that we couldn't leave out. The proof is by considering the properties of lattice polygons, like Pick's Theorem. Let's prove the if direction first. Set the points (a, b) and (c, d) in the coordinate plane, and consider the polygon P with vertices $(0, 0)$, (a, b), and (c, d). What is so special about P? To start, we have the following:

Claim 1.4. *P has no lattice points in its interior.*

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Proof. Suppose, by contradiction, that there is a lattice point in the interior. Then the y-coordinate of this lattice point must be less than the greater of b and d. Now, note that the multiplicative inverse of the slope of the line passing through the origin and (a, b) is $\frac{a}{b}$. Similarly the multiplicative inverse of the slope of the line passing through the origin and (c, d) is $\frac{c}{d}$, and something similar holds for our interior lattice point (m, n). Since (m, n) is bounded by the line connecting the origin and (a, b) and the line connecting the origin and (c, d), a property of slopes gives us

$$\frac{d}{c} < \frac{n}{m} < \frac{b}{a}.$$

This is equivalent to

$$\frac{a}{b} < \frac{m}{n} < \frac{c}{d}.$$

However, this contradicts the fact that $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in F_n because either $n < b$ or $n < d$, making $\frac{m}{n}$ part of $F_{\max(b,d)}$ if $\gcd(m, n) = 1$. (If the greatest common divisor is not equal to 1, then there must be an interior lattice point on the line segment connecting the origin and (p_3, q_3) that is closer to the origin than (p_3, q_3), so we can repeat this argument with that interior lattice point instead.) Therefore, there are no lattice points in the interior of P. ∎

In fact, we can say even more:

Claim 1.5. *The only boundary points of P are the vertices of the triangle.*

Proof. Obviously there are no lattice points in the interior of the segments connecting the origin and each of (a, b) and (c, d). Therefore, it suffices to show that there are no lattice points in the interior of the segment connecting (a, b) and (c, d). Suppose that there were a lattice point (p, q) in the interior. Then from what we know about slopes, we get that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$. Note that $q \le \max(b, d)$, so $\frac{p}{q} \in F_{\max(b,d)}$, which is a contradiction. ∎

To finish the proof of the if part, we can say by Pick's Theorem that the area of P is $\frac{1}{2}$. However, by linear algebra, the area of P is $\frac{bc-ad}{2}$ since $\frac{a}{b} < \frac{c}{d}$. This means that $bc - ad = 1$, proving the if part.

To prove the only if part, assume that $bc - ad = 1$. Then the area of P is $\frac{1}{2}$, which by Pick's Theorem immediately implies that P has no interior lattice points and no boundary lattice points other than its vertices. Now consider the parallelogram Q formed by the points $(0, 0), (a, b), (c, d)$, and $(a + c, b + d)$. Then Q has area 1 and contains all the points (x, y) between the line segments connecting the origin and each of (a, b) and (c, d) such that $y \le \max(b, d)$. Because Q is just two copies of P, Q has no interior lattice points or boundary lattice points other than the origin, so no such point (x, y) exists. This means that $\frac{a}{b}$ and $\frac{c}{d}$ are indeed adjacent in a Farey sequence. This completes the proof.

Corollary 1.6. *If $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent in $F_{\max(b,d)}$, then they are adjacent in all F_k, where $\max(b, d) \le k \le b + d - 1$. Moreover, in F_{b+d}, the unique term in between $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.*

Proof. Suppose that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ are adjacent Farey fractions. Then $pb - aq = 1$, $qc - pd = 1$, and $bc - ad = 1$. Multiplying the first equation by c and the second by a, we get $pbc - acq = c$ and $acq - pda = a$. Adding these two equations together, we get $p(bc - ad) = p = a + c$. Solving back for q gives $q = b + d$. ∎

So in fact, we can generate the Farey sequences as far as we want to by continuously adding this mediant fraction $\frac{a+c}{b+d}$ in between any two terms $\frac{a}{b}$ and $\frac{c}{d}$.

Proposition 1.7. *If $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent terms in a Farey sequence F_n,, then $b+d>n$.*

Proof. Very straightforward. If $b+d \leq n$, then the fraction $\frac{a+c}{b+d}$, which is in between these two fractions, is in F_n, which contradicts the fact that the two fractions are adjacent. ■

Finally, we present one last important but strange theorem related to the size of the Farey sequence.

Theorem 1.8. *The size of the Farey sequence $|F_n|$ approaches $\frac{3n^2}{\pi^2}$ as n gets large.*

Proof. By Theorem 1.2, we know that $|F_n| = 1 + \sum_{i=1}^{n} \phi(n)$. Therefore, since $\frac{1}{n^2}$ becomes 0 as n approaches infinity, we'd like to show that

$$\frac{\phi(1)+\phi(2)+\cdots+\phi(n)}{n^2}$$

approaches $\frac{3}{\pi^2}$ as n approaches infinity.

To do so, we will prove that the probability that two integers chosen at random between 1 and n inclusive are relatively prime approaches $\frac{6}{\pi^2}$ as n approaches infinity. Then we will show using a different method that this probability is $\frac{2(\phi(1)+\phi(2)+\cdots+\phi(n))-1}{n^2}$, which implies the result.

Let's first show that the probability that two integers a, b chosen at random between 1 and n inclusive are relatively prime approaches $\frac{6}{\pi^2}$ as n approaches infinity. First assume that n is large enough so that all residue classes modulo a prime are equally likely to be chosen. Select an arbitrary prime p. Then p cannot be in the prime factorization of both a and b. The probability that a is divisible by p is $\frac{1}{p}$, and the probability that b is divisible by p is also $\frac{1}{p}$. Hence by complementary counting the probability that $\gcd(a, b)$ does not have a factor of p is $1 - \frac{1}{p^2}$. Multiplying this for all primes p, we get

$$p = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots.$$

If we invert this, we get

$$\frac{1}{p} = \frac{1}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots}.$$

However, we now recognize the right-hand side to be $\zeta(2) = \frac{\pi^2}{6}$ where ζ denotes the Riemann Zeta function. Therefore, $p = \frac{6}{\pi^2}$.

An alternate way of finding the probability is to just use casework. Let x_n be the number of ways to choose integers $1 \leq a, b \leq n$ such that $\gcd(a, b) = 1$. If $1 \leq a, b \leq n-1$, there are x_{n-1} ways to choose such a, b. Now suppose that $a = n$. Note that $b \neq n$, so there are $\phi(n)$ choices for b. Similarly, if $b = n$, then there are $\phi(n)$ choices for a. Hence $x_n = x_{n-1} + 2\phi(n)$. Noting that $x_1 = 1$, we get that

$$x_n = 2(\phi(1)+\phi(2)+\cdots+\phi(n)) - 1.$$

The probability that $\gcd(a,b)=1$ is therefore

$$\frac{2(\phi(1)+\phi(2)+\cdots+\phi(n))-1}{n^2}.$$

As n approaches ∞, we get that this approaches $\frac{6}{\pi^2}$. Plugging in, we get that

$$\frac{2(\phi(1)+\phi(2)+\cdots+\phi(n))}{n^2}\approx\frac{6}{\pi^2}.$$

Therefore,

$$\frac{\phi(1)+\phi(2)+\cdots+\phi(n)}{n^2}\approx\frac{|F_n|}{n^2}\approx\frac{3}{\pi^2},$$

which completes the proof. ■

Now that we have talked a bit about Farey sequences and their properties, we can now move onto applications of them.

2. Applications to Rational Approximations

The first application is used to approximate irrational numbers, as shown below:

Theorem 2.1. ***(Dirichlet's Theorem on Rational Approximations)*** *If α is a real number in in $[0,1]$ and if n is a positive integer, then there is a rational number h/k with $0<k\le n$ such that*

$$\left|\alpha-\frac{h}{k}\right|\le\frac{1}{k(n+1)}.$$

Proof. Let $\frac{a}{b}$ and $\frac{c}{d}$ are the fractions in F_n such that $\frac{a}{b}<\alpha<\frac{c}{d}$. We claim that we can either set $\frac{h}{k}=\frac{a}{b}$ or $\frac{h}{k}=\frac{c}{d}$.

Suppose, for the sake of contradiction, that neither $\frac{a}{b}$ nor $\frac{c}{d}$ satisfy this property; that is,

$$\left|\alpha-\frac{a}{b}\right|>\frac{1}{b(n+1)}$$

and

$$\left|\alpha-\frac{c}{d}\right|>\frac{1}{d(n+1)}.$$

We can eliminate both absolute values using the fact that $\frac{a}{b}\le\alpha\le\frac{c}{d}$ to get

$$\alpha-\frac{a}{b}>\frac{1}{b(n+1)}$$

and

$$\frac{c}{d}-\alpha>\frac{1}{d(n+1)}.$$

Solving for α in each inequality, we get

$$\alpha>\frac{a}{b}+\frac{1}{b(n+1)}$$

and

$$\alpha<\frac{c}{d}-\frac{1}{d(n+1)}.$$

Hence we can deduce that

$$\frac{a}{b} + \frac{1}{b(n+1)} < \frac{c}{d} - \frac{1}{d(n+1)}.$$

Rearranging gives

$$\frac{1}{b(n+1)} + \frac{1}{d(n+1)} < \frac{c}{d} - \frac{a}{b}.$$

Combining fractions tells us that

$$\frac{b+d}{bd(n+1)} < \frac{bc-ad}{bd}.$$

However, we know from Theorem 1.3 that $bc - ad = 1$, so

$$\frac{b+d}{bd(n+1)} < \frac{1}{bd}.$$

Multiplying both sides by $bd(n+1)$, we get

$$b + d < n + 1.$$

However, this contradicts Proposition 1.7, which states that $b + d > n$.

Therefore, we can either set $\frac{h}{k} = \frac{a}{b}$ or $\frac{h}{k} = \frac{c}{d}$. ■

This surprising result not only demonstrates how we can get arbitrarily close to an irrational number using rational approximations as we want to, but it shows that Farey sequences do the job for us. This is one of many examples that show how powerful Farey sequences can be.

Example. Let's use Farey sequences to approximate $\sqrt{2} - 1$. Some decent approximations are that $\frac{0}{1} < \sqrt{2} - 1 < \frac{1}{2}$. To get the next Farey fraction, we can find the mediant:

$$\frac{1}{3} < \sqrt{2} - 1 < \frac{1}{2}.$$

We can again take the mediant:

$$\frac{2}{5} < \sqrt{2} - 1 < \frac{1}{2}.$$

Ad once again:

$$\frac{2}{5} < \sqrt{2} - 1 < \frac{3}{7}.$$

And finally one last time:

$$\frac{2}{5} < \sqrt{2} - 1 < \frac{5}{12}.$$

When we add 1 to both sides, we get

$$\frac{7}{5} < \sqrt{2} < \frac{17}{12}.$$

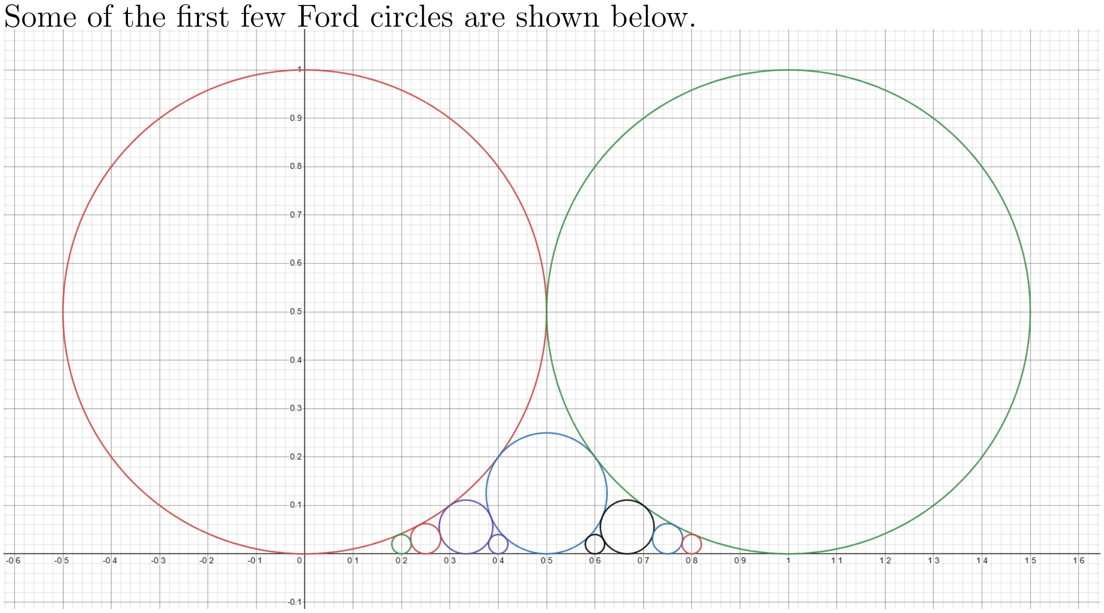
Both fractions end up serving as good approximations to $\sqrt{2}$.

3. Applications to Ford Circles

We first define what a Ford Circle is:

Definition 3.1. For every rational number p/q in lowest terms, the Ford Circle $C(p,q)$ is the circle with center $(\frac{p}{q}, \frac{1}{2q^2})$ and radius $\frac{1}{2q^2}$.

Some of the first few Ford circles are shown below.



We notice that a lot of them are tangent to each other, or don't intersect. None of them intersect at 2 points. In fact, we can say something even stronger, which is how Ford circles are connected to Farey sequences:

Theorem 3.2. *The representative Ford circles of two distinct fractions $\frac{p}{q}$ and $\frac{r}{s}$ are either tangent at one point or do not intersect at all. They are tangent if and only if $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent in some Farey sequence.*

Proof. Without loss of generality, assume that $\frac{p}{q} > \frac{r}{s}$. We will show that the distance between their centers is greater than or equal to the sum of the radii with equality when $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent in a Farey sequence, implying the result. We have that the centers of the two circles are $\left(\frac{p}{q}, \frac{1}{2q^2}\right)$ and $\left(\frac{r}{s}, \frac{1}{2s^2}\right)$. We know that the sum of the radii is

$$\frac{1}{2q^2} + \frac{1}{2s^2},$$

and we also know that the distance between the two centers is

$$\sqrt{\left(\frac{p}{q} - \frac{r}{s}\right)^2 + \left(\frac{1}{2q^2} - \frac{1}{2s^2}\right)^2}.$$

Therefore, we wish to show that

$$\frac{1}{2q^2} + \frac{1}{2s^2} \leq \sqrt{\left(\frac{p}{q} - \frac{r}{s}\right)^2 + \left(\frac{1}{2q^2} - \frac{1}{2s^2}\right)^2}.$$

Squaring both sides, we get

$$\left(\frac{1}{2q^2}+\frac{1}{2s^2}\right)^2 \le \left(\frac{p}{q}-\frac{r}{s}\right)^2+\left(\frac{1}{2q^2}-\frac{1}{2s^2}\right)^2.$$

Subtracting $\left(\frac{1}{2q^2}-\frac{1}{2s^2}\right)^2$ from both sides and using Difference of Two Squares, we get

$$\frac{1}{q^2s^2} \le \left(\frac{p}{q}-\frac{r}{s}\right)^2.$$

Therefore, square-rooting both sides, we get

$$\frac{p}{q}-\frac{r}{s}=\frac{ps-qr}{qs}\ge\frac{1}{qs},$$

so we wish to show that

$$ps-qr\ge 1.$$

However, since $\frac{p}{q}>\frac{r}{s}$, we know that $ps-qr>0$, which is equivalent to $ps-qr\ge 1$ when p,q,r,s are integers. Equality holds if and only if $ps-qr=1$, which by Theorem 1.3 is equivalent to $\frac{p}{q}$ and $\frac{r}{s}$ being adjacent in some Farey sequence. ■

Another natural question to ask is what the total area of all the Ford circles with $p\le q$ is. It turns out that we have the following:

Theorem 3.3. *The area of all the Ford Circles satisfying $p\le q$ is $\frac{\pi^2\zeta(3)}{4\zeta(4)}$, where $\zeta(x)$ is the Riemann zeta function.*

To prove this, we will need the following:

Claim 3.4. *If $\phi(n)$ denotes Euler's Totient function, then*

$$\sum_{q=1}^{\infty}\frac{\phi(q)}{q^4}=\frac{\zeta(3)}{\zeta(4)}.$$

Proof. Write $q=p_1^{e_1}\cdot p_2^{e_2}\cdots p_k^{e_k}$, where the p_i are primes. Then

$$\phi(q)=q\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_k}\right).$$

Plugging in, we get

$$\frac{\phi(q)}{q^4}=\frac{\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)}{p_1^{3e_1}\cdot p_2^{3e_2}\cdots p_k^{3e_k}}.$$

Now consider a particular prime p. Then for all multiples of p, we have the factor

$$\left(1-\frac{1}{p}\right)\cdot\left(\frac{1}{p^3}+\frac{1}{p^6}+\frac{1}{p^9}+\dots\right).$$

Evaluating the infinite series, we get

$$\left(1-\frac{1}{p}\right)\cdot\frac{1}{p^3-1}=\frac{p-1}{p(p^3-1)}.$$

However, we need to add 1 to this to account for the non-multiples of p. Doing so, we get

$$\frac{p-1}{p(p^3-1)}+1=\frac{p^4-1}{p(p^3-1)}=\frac{1-\frac{1}{p^4}}{1-\frac{1}{p^3}}=\frac{\frac{1}{1-\frac{1}{p^3}}}{\frac{1}{1-\frac{1}{p^4}}}.$$

Multiplying over all primes, we get our infinite series to evaluate to

$$\frac{\Pi\frac{1}{1-\frac{1}{p^3}}}{\Pi\frac{1}{1-\frac{1}{p^4}}}.$$

The numerator is well-known to be $\zeta(3)$, and the denominator is similarly well-known to be $\zeta(4)$. Therefore, the series converges to $\frac{\zeta(3)}{\zeta(4)}$, proving the claim. ■

Remark 3.5. In fact, we can generalize the above argument to get that

$$\sum_{q=1}^{\infty}\frac{\phi(q)}{q^s}=\frac{\zeta(s-1)}{\zeta(s)}$$

for any $s\in\mathbb{C}$ with real part greater than 2.

Now let's prove Theorem 3.3 using this result.

Proof. Let's consider $C(p,q)$ for $\gcd(p,q)=1$. Fixing q, we get that there are $\phi(q)$ options for p, and each circle has area $\pi\left(\frac{1}{2q^2}\right)^2=\frac{\pi^2}{4}\cdot\frac{1}{q^4}$. Hence the area of all the Ford circles with $p\leq q$ for a particular q is $\frac{\pi^2}{4}\cdot\frac{\phi(q)}{q^4}$. Summing over all q and using Claim 3.4, we get that the total area of the Ford Circles is

$$\frac{\pi^2}{4}\cdot\sum_{q=1}^{\infty}\frac{\phi(q)}{q^4}=\frac{\pi^2\zeta(3)}{4\zeta(4)},$$

as desired. ■

4. Conclusion

In this paper, we have established some of the key properties of the Farey sequence. These properties include its size in terms of Euler's totient function, the $bc-ad=1$ property that we ended up proving using lattice points, the mediant property which allows us to construct Farey sequences for as long as we want, and the fact that $|F_n|\sim\frac{3n^2}{\pi^2}$.

We then moved onto key applications. We showed that given any positive integer n and any real number α, we can use Farey sequences to approximate α to a precision of at most $\frac{1}{n+1}$. We discussed Ford circles and how $C(p,q)$ and $C(r,s)$ are adjacent if and only if $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent in some Farey sequence. We concluded by finding the area of all the Ford circles in terms of the Riemann zeta function. This could end up leading to a possible connection to the Riemann Hypothesis, something that could be further explored.

5. Acknowledgements

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6. References

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