FAREY SEQUENCES

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ABSTRACT. In this article, we discuss the multiple algebraic properties of Farey Sequences. We link the notion of Farey Sequences with the study of Diophantine Approximation. We furthermore present several geometric interpretations of Farey Fractions.

1. Introduction

The name Farey fractions [1] comes from the English geoligist John Farey Sr., who wrote a letter about these sequences that was published in the *Philosophical Magazine* in 1816. Interestingly enough, another mathematician, Charles Haros, independently published similar results in 1802, which was not known to Farey. Thus, the Farey Sequence is sometimes known as Haros-Farey Sequence.

The concept of Farey fractions is not too difficult to understand, and the assumed knowledge for the following paper is a thorough understanding of algebra and geometry.

In this paper we'll discuss and prove some interesting results that the Farey Fractions yield and their relationship to a topic known as Diophantine Approximation. We'll also cover some nice geometric interpretations of the Farey Sequences.

2. Definitions and Properties

Definition 2.1. We define the **Farey Sequence** of order n, denoted F_n, as the sequence of rational numbers of the form $\frac{a}{b}$, where a and b are positive integers such that $1 \le a \le b \le n$ and $\gcd(a, b) = 1$ (reduced form). We write the Farey Sequence in increasing order, starting from $\frac{0}{1}$ and ending with $\frac{1}{1}$.

Example. The Farey Sequence of order 3 is the sequence of numbers

$$\left(\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right).$$

Due to the bounds on Farey fractions, we'll assume for the rest of the paper that fractions are in the range $[0, 1]$, unless otherwise specified, and that all fractions are in reduced form.

We can deduce one elementary property of the Farey Sequence.

Corollary 2.2. The number of elements in the Farey Sequence F_n is $(\sum_{i=1}^{n} \phi(n)) + 1$, where $\phi(n)$ is the *Euler Totient Function*, which counts the number of integers up to n relatively prime to n.

Proof: The +1 accounts the fraction $\frac{0}{1}$; each element $\phi(i)$ of the summation accounts for the number of fractions in reduced form with denominator i in the interval $[0,1]$. □

More complicated questions arise when talking about the mechanics of the Farey Sequences. For example, given two rational numbers in the interval $[0,1]$, are they ever consecutive in some Farey Sequence?

Theorem 2.3. Two distinct rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in F_k, where $k = \max(b,d)$, if and only if $ad - bc = \pm 1$.

We'll prove this in a few parts. First, we introduce the notion of the **mediant** of two fractions.

Definition 2.4. The mediant of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is defined as $\frac{a+c}{b+d}$.

The mediant of two fractions offers an interesting property if these two fractions happen to be consecutive in some Farey Sequence.

Proposition 2.5. $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof: Observe that

$$\frac{a}{b} < \frac{a+c}{b+d} \iff ab + ad < ab + bc \iff ad < bc,$$

which follows from $\frac{a}{b} < \frac{c}{d}$. On the other side,

$$\frac{a+c}{b+d} < \frac{c}{d} \iff ad + dc < bc + cd \iff ad < bc,$$

in a similar fashion. □

Corollary 2.6. For consecutive Farey Fractions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a+c}{b+d}$ is in reduced form.

Proof: Observe that $b + d \leq 2\max(b,d)$. Since $\frac{a+c}{b+d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$ in some Farey Sequence, the fraction $\frac{a+c}{b+d}$ in reduced form cannot have denominator less than or equal to $\max(b,d)$. Suppose for contradiction's sake that $\gcd(a+c, b+d) = k > 1$. Then

$$\frac{a+c}{b+d} = \frac{\frac{a+c}{k}}{\frac{b+d}{k}}.$$

But $\frac{b+d}{k} \leq \frac{b+d}{2} \leq \max(b,d)$, a contradiction to the consecutive nature of $\frac{a}{b}$ and $\frac{c}{d}$, hence $\frac{a+c}{b+d}$ is in reduced form. □

The key idea is that each fraction is the mediant of two other consecutive fractions of a previous sequence. This is important to proving many properties of the Farey Sequences. For example, now we can go ahead and prove **Theorem 2.3**:

Proof of Theorem 2.3. We can proceed by induction. Checking the cases $n = 1, 2, 3$ are left as an exercise to the reader.

In the forwards direction, take any two Farey Fractions $\frac{a}{b}$ and $\frac{c}{d}$; then

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

Suppose $ad - bc = 1$ and there exists a fraction $\frac{e}{f}$ with $f \leq \max(b, d)$ and $\frac{a}{b} > \frac{e}{f} > \frac{c}{d}$. Then,

$$\frac{af - be}{bf} \geq \frac{1}{be} \geq \frac{1}{bd} = \frac{a}{b} - \frac{c}{d},$$

a contradiction. The proof when $ad - bc = -1$ is identical.

For the reverse direction, suppose that this is true for all Farey Sequences F_k where $k < n$. Then, all Farey fractions are of the form $\frac{a+c}{b+d}$ where $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in $F_{\max(b,d)}$. Thus, if we have two consecutive fractions in F_n, they must be of the form $\frac{a}{b}$ and $\frac{a+c}{b+d}$ where $\frac{a}{b}$ and $\frac{c}{d}$ were consecutive fractions in some F_i where $i < n$.

Now, note that $a(b + d) - b(a + c) = ad - bc = \pm 1$ from the inductive hypothesis, hence proving the inductive claim. □

There is a nice geometric interpretation of the mediant of two consecutive Farey fractions; we present it below.

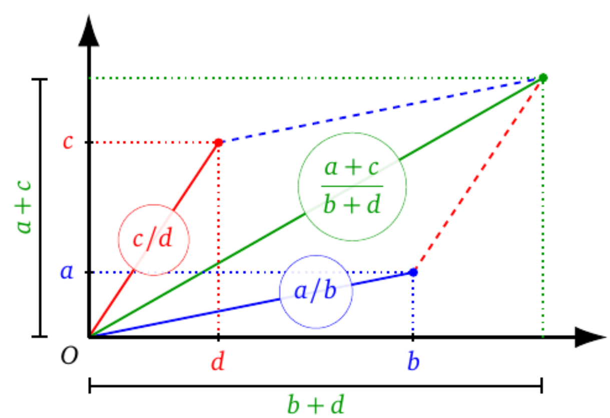

Figure 1. The mediant of two Farey Fractions [2]

The idea is that two consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ can be represented as the vectors $u = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v = \begin{bmatrix} c \\ d \end{bmatrix}$. The condition that $\frac{a+c}{b+d}$ is the first fraction to come between $\frac{a}{b}$ and $\frac{c}{d}$ is then represented by the fact that no lattice points are present inside the parallelogram above. Note that if a lattice point is inside the parallelogram, then there exists a fraction between $\frac{a}{b}$ and $\frac{c}{d}$; moreover, this point has y coordinate less than $b + d$, implying that a fraction arises before $\frac{a+c}{b+d}$. Hence, no lattice points exist inside the parallelogram.

In fact, this gives rise to an alternate proof of Theorem 2.3 [3]. From Pick's Theorem, the area of the parallelogram with sides being the vectors u and v is equal to $I+\frac{B}{2}-1$, where I is the number of interior points in the parallelogram and B is the number of boundary lattice points of the paralllelogram. Note that $I=0$ and $B=4$ for consecutive Farey Fractions; $B=4$ arises from the vertices of the parallelogram, and we've already seen that $I=0$. Hence, $A=0+4/2-1=1$.

At the same time, from the Shoelace Formula, the area of such a parallelogram is $|ad-bc|$, hence $|ad-bc|=1$.

In summary:

Corollary 2.7. If no integer lattice points exist strictly inside the paralellogram generated by the vectors $u=\begin{bmatrix}a\\b\end{bmatrix}$ and $v=\begin{bmatrix}c\\d\end{bmatrix}$, where $a<b$ and $c<d$, then $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in the Farey Sequences $F_{\max(b,d)}$ up until F_{b+d}.

3. Diophantine Approximation

The central idea around this is approximating irrational numbers with rational numbers. From the denseness of the rational numbers, we know that, for every real number x and $\epsilon>0$, there exists integer p,q with

$$\left|\frac{p}{q}-x\right|<\epsilon.$$

As an example, we'll use

$$\sqrt{2}=1.4142135\ldots$$

We could get approximations by taking fractions like $\frac{14142}{10000}=\frac{7071}{5000}$, but this is boring! The idea of Diophantine Approximation is to minimize the denominator while getting as close as possible to the value at hand.

Theorem 3.1. *(Weak Diophantine Approximation.)* For every positive irrational number x and $\epsilon>0$, there exist infinitely many $p,q\in\mathbb{Z}$ with $\left|\frac{p}{q}-x\right|<\frac{1}{2q}$.

Proof: Bound x between any two rational numbers with the same denominator, i.e. choose p_1,q_1 such that

$$\frac{p_1}{q_1}<x<\frac{p_1+1}{q_1}.$$

Then choose the fraction x is closer to; since the sum of the distances from x to both fractions adds to $\frac{1}{q_1}$, it follows that either $\left|\frac{p_1}{q_1}-x\right|\le\frac{1}{2q_1}$ or $\left|\frac{p_1+1}{q_1}-x\right|\le\frac{1}{2q_1}$. In fact, this bound is tight because x is irrational.

The choice of p_1 and q_1 can be made for any positive integer q_1, hence infinitely many such approximations exist. □

This bound is a little weak, however, as evidenced by the fact that it can be done for all choices of q. If we take a look at the example of $\sqrt{2}$ again, then $\frac{99}{70}$ is a very tight approximation for $\sqrt{2}$; in particular,

$$|\sqrt{2} - \frac{99}{70}| = 0.00007215191 \approx 7 \cdot 10^{-5}.$$

This looks more like an approximation of $\frac{1}{q^2}$ or $\frac{1}{2q^2}$ distance away; how close can we get?

In Theroem 3.1, we bounded x by two fractions of the same denominator. Can we find a better bound, though? The answer is through Farey Fractions.

Theorem 3.2. For every positive irrational number x and $\epsilon > 0$, there exist infinitely many $p, q \in \mathbb{Z}$ with $\left|\frac{p}{q} - x\right| < \frac{1}{2q^2}$.

Proof: Without loss of generality, assume $0 < x < 1$. (Otherwise, we can just shift x to be in this interval.) Now, in some Farey Sequence F_n, bound x by the two consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$, with $\frac{a}{b} < \frac{c}{d}$. We claim that either

$$\left|\frac{a}{b} - x\right| < \frac{1}{2b^2}$$

or

$$\left|\frac{c}{d} - x\right| < \frac{1}{2d^2}.$$

Suppose for contradiction that neither of these are true. Then we must have

$$\left|\frac{a}{b} - x\right| + \left|\frac{c}{d} - x\right| \geq \frac{1}{2b^2} + \frac{1}{2d^2}.$$

At the same time, we also have

$$\left|\frac{a}{b} - \frac{c}{d}\right| = \left|\frac{ad - bc}{bd}\right| = \left|\frac{1}{bd}\right| = \frac{1}{bd}$$

following from **Theorem 2.3**. Now observe that

$$\left|\frac{a}{b} - x\right| + \left|\frac{c}{d} - x\right| = x - \frac{a}{b} + \frac{c}{d} - x = \frac{c}{d} - \frac{a}{b}.$$

Thus, we must have

$$\frac{1}{2b^2} + \frac{1}{2d^2} \leq \frac{1}{bd} \iff 2bd \geq b^2 + d^2 \iff 0 \geq (b - d)^2.$$

This is a contradiction as $b \neq d$; no two consecutive Farey Fractions apart from $\frac{0}{1}$ and $\frac{1}{1}$ have the same denominator. This follows from Theorem 2.3; if $b = d$, then $b(a - c) = 1$, which is impossible for integers a, b, c unless $b = 1$. Thus, at least one of the inequalities holds, implying that we've found an approximation.

In fact, as n grows larger, the fractions generated by $\frac{a}{b}$ and $\frac{c}{d}$, i.e. $\frac{a+c}{b+d}$ and so forth, will split the consecutive fraction interval in which x is in infinitely many times; thus, infinitely many such approximations exist for x. $\square$

Do there exist better constants c apart from 2 for $\frac{1}{cq^2}$? As it turns out, there do. The proof once again follows from Farey Fractions!

Theorem 3.3. (*Hurwitz's Theorem.*) For every positive irrational number x and $\epsilon > 0$, there exist infinitely many $p, q \in \mathbb{Z}$ with $\left|\frac{p}{q} - x\right| < \frac{1}{\sqrt{5}q^2}$.

Proof: We'll follow an algebraic proof of Zukin's [4].

In particular, we'll show that the above inequality is satisfied by one of the fractions

$$\frac{a}{b}, \frac{c}{d}, \frac{a+c}{b+d}$$

where $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in some Farey Sequence.

In particular, assume without loss of generality that x is bounded by $\frac{a+c}{b+d}$ and $\frac{c}{d}$. For contradiction purposes, assume that

$$x - \frac{a}{b} \geq \frac{1}{\sqrt{5}b^2},$$

$$x - \frac{a+c}{b+d} \geq \frac{1}{\sqrt{5}(b+d)^2},$$

$$\frac{c}{d} - x \geq \frac{1}{\sqrt{5}d^2}.$$

For simplicity sake, let $a + c = e$ and $b + d = f$.

Adding the first and third inequalities yields

$$\frac{c}{d} - \frac{a}{b} \geq \frac{1}{\sqrt{5}b^2} + \frac{1}{\sqrt{5}d^2},$$

and adding the second and third inequalities yields

$$\frac{c}{d} - \frac{e}{f} \geq \frac{1}{\sqrt{5}d^2} + \frac{1}{\sqrt{5}f^2}.$$

Since $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in some Farey Sequence as is $\frac{e}{f}$ and $\frac{c}{d}$, we can write

$$\frac{1}{bd} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b^2} + \frac{1}{d^2}\right),$$

$$\frac{1}{df} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{d^2} + \frac{1}{f^2}\right).$$

Clearing the denominators gives us

$$bd\sqrt{5} \geq b^2 + d^2,$$

$$df\sqrt{5} \geq d^2 + f^2.$$

Now, adding the two yields

$$d\sqrt{5}(b + f) = d\sqrt{5}(2b + d) \geq b^2 + 2d^2 + f^2 = 2b^2 + 3d^2 + 2bd,$$

which can be rewritten as

$$0 \geq \frac{1}{2}((\sqrt{5}-1)d - 2b)^2.$$

Thus,

$$2b = d(1-\sqrt{5}) \iff \sqrt{5} = 1 - \frac{2b}{d},$$

a contradiction because $\sqrt{5}$ is irrational. Thus, at least one of the three original inequalities holds.

As we take fractions from more advanced Farey Sequences the fractions that bound x change, generating infinitely many approximations $\frac{p}{q}$, as desired. □

It has been proven $\sqrt{5}$ is the optimal constant possible [5]; other constants generate only finitely many approximations for select values of x. For example, we saw that $\sqrt{2}$ is approximated by $\frac{99}{70}$, the approximation of which yields a better constant than $\sqrt{5}$ (approximately 2.82); however, only finitely many such fractions exist with that constant. In fact, for $x = \sqrt{2}$ specifically, we can show that no approximations exist for $c = 3$. But we digress...

4. Ford Circles

The best way to start this off is with a picture.

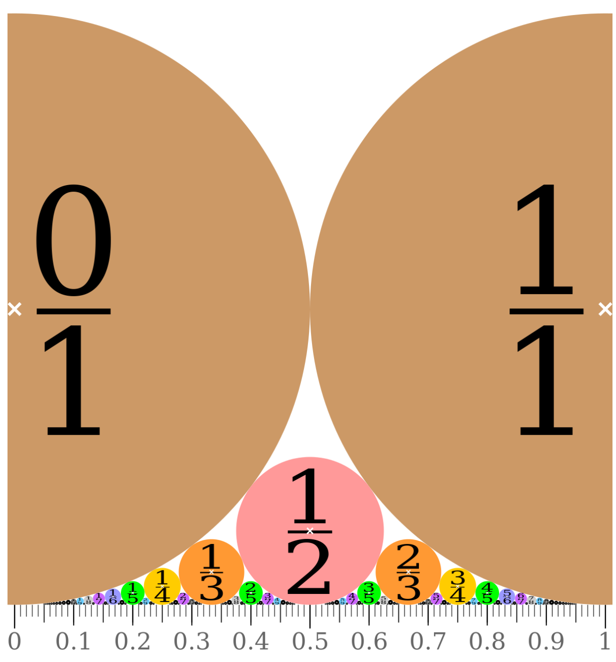

Figure 2. A picture is worth a thousand words. [1]

How do we construct the above figure?

Construction. For each $\frac{a}{b}$ in F_n, draw the circle centered at $\left(\frac{a}{b}, \frac{1}{2b^2}\right)$ with radius $\frac{1}{2b^2}$. (Do this as n approaches ∞.)

The main observation to be made from this picture is that two circles appear to be tangent if and only if their corresponding Farey Fractions are consecutive in some Farey Sequence. This is the main result following from Ford Circles.

Theorem 4.1. Take two Farey Fractions $\frac{a}{b}$ and $\frac{c}{d}$, and consider the circles generated by the above construction. The circles are externally tangent if and only if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in some F_n.

Proof: For two circles to be tangent to each other, the sum of their radii must be equal to the distance between their centers. In other words, we must have

$$\frac{1}{2b^2} + \frac{1}{2d^2} = \sqrt{\left(\frac{a}{b} - \frac{c}{d}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2}.$$

Squaring both sides and rearranging, this is equivalent to

$$\left(\frac{a}{b} - \frac{c}{d}\right)^2 = \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2 - \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2 = \frac{1}{b^2d^2}.$$

Thus, we must have

$$\left|\frac{a}{b} - \frac{c}{d}\right| = \frac{1}{bd} \iff |ac - bd| = 1,$$

which is **Theorem 2.3**. The theorem is bidirectional, so we reach our desired conclusion. □

5. References

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