Farey Sequences and the Stern–Brocot Tree

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1 Introduction

This expository paper concerns Farey sequences and the closely-related Stern–Brocot tree. These two objects describe rational numbers with bounded denominators in increasing order. From the Farey sequence ordering and the Stern–Brocot tree, we discover a slew of elegant properties relating to continued fractions and mediants. We will begin by describing Farey sequences and proving key properties about them. We will then introduce the Stern–Brocot tree and discuss how it relates to Farey sequences. Finally, we will apply these methods to rational approximation.

2 Farey Sequences

In this section, we will define Farey sequences, prove their key properties, and show how to construct them.

Definition 2.1. Given an integer m, the m^{th} *Farey sequence* F_m is the increasing sequence of reduced fractions with numerator and denominator less than or equal to m, with the numerator less than or equal to the denominator.

For example, the 5^{th} Farey sequence is

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

Farey sequences can be visualized by drawing their *sunburst* (see Figure 1).

This geometric object represents the various fractions as points of the form (d, n). Such a geometric interpretation leads to our first proposition regarding Farey sequences.

Proposition 2.2. *If $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ are two consecutive elements of a Farey sequence, $n_2d_1 - n_1d_2 = 1$.*

Proof. As shown in Figure 1, consider the triangle formed by $(0,0)$, (d_1, n_1), and (d_2, n_2). Then, linear algebra tells us that $A = \frac{1}{2}(d_1, n_1) \times (d_2, n_2) = \frac{1}{2}(n_2d_1 - n_1d_2)$. Now, consider the geometric interpretation of $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ being consecutive. Note that $\frac{n_1}{d_1}$ (resp., $\frac{n_2}{d_2}$) is the slope of the line connection $(0,0)$ to (d_1, n_1) (resp., (d_2, n_2)). Thus, no line with slope between $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ passing through $(0,0)$ passes through any other lattice point in the square $0 \le x, y \le m$. Hence, our triangle contains no interior lattice points. Thus, by Pick's theorem, the area of the triangle is $\frac{1}{2}$, so $n_2d_1 - n_1d_2 = 1$. □

For example, in F_5, the element $\frac{1}{3}$ is immediately followed by $\frac{2}{5}$, and $3 \times 2 - 1 \times 5 = 1$.

Applying this proposition allows us to relate an entry of a Farey sequence to the two adjacent entries.

Proposition 2.3. *If $\frac{n_1}{d_1}, \frac{n_2}{d_2}, \frac{n_3}{d_3}$ are consecutive entries of a Farey sequence in reduced form, $\frac{n_2}{d_2} = \frac{n_1+n_3}{d_1+d_3}$. In other words, $\frac{n_2}{d_2}$ is the* mediant *of $\frac{n_1}{d_1}$ and $\frac{n_3}{d_3}$.*

Proof. By Proposition 2.2, $n_2d_1 - d_2n_1 = n_3d_2 - n_2d_3 = 1$. Working with just the first equality, we find that $n_2(d_1 + d_3) = d_2(n_1 + n_3)$, and so $\frac{n_2}{d_2} = \frac{n_1+n_3}{d_1+d_3}$ as claimed. □

For example, if we consider the three adjacent elements $\frac{3}{5}, \frac{2}{3}, \frac{3}{4}$ in F_5, we may note that $\frac{3+3}{5+4} = \frac{6}{9} = \frac{2}{3}$.

Expressing Farey sequences in terms of the mediant allows us to find a formula for the next element of a Farey sequence.

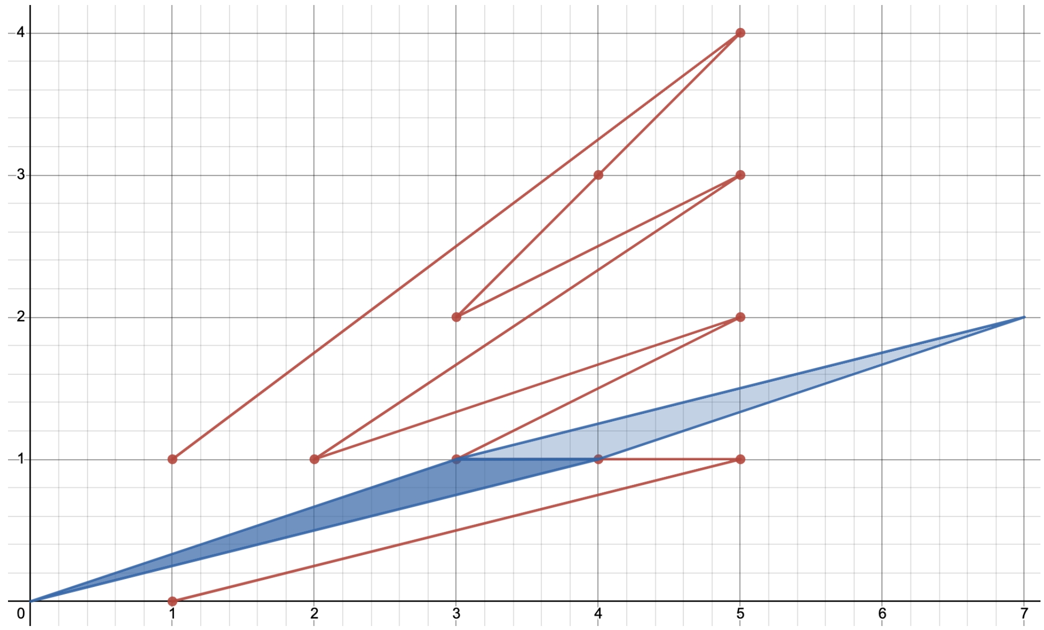

Figure 1: A geometric visualization of F_5 and a parallelogram representing $(4,1) \times (3,1)$.

Proposition 2.4. *Considering F_m, if two consecutive values are $\frac{n_1}{d_1}, \frac{n_2}{d_2}$, the next value is*

$$\frac{n_3}{d_3} = \frac{\left\lfloor \frac{m+d_1}{d_2} \right\rfloor n_2 - n_1}{\left\lfloor \frac{m+d_1}{d_2} \right\rfloor d_2 - d_1}.$$

Proof. We have $\frac{n_2}{d_2} = \frac{n_1+n_3}{d_1+d_3}$, or, equivalently, $kn_2 = n_1 + n_3, kd_2 = d_1 + d_3$. Then, $n_3 = kn_2 - n_1, d_3 = kd_2 - d_1$, and so

$$\begin{aligned} \frac{n_3}{d_3} &= \frac{kn_2 - n_1}{kd_2 - d_1} \\ &= \frac{n_2}{d_2} + \frac{d_2(kn_2 - n_1) - n_2(kd_2 - d_1)}{d_2(kd_2 - d_1)} \\ &= \frac{n_2}{d_2} + \frac{1}{d_2(kd_2 - d_1)} \end{aligned}$$

This makes it clear that $\frac{n_3}{d_3}$ decreases as k increases. Hence, the next element of the Farey sequence can be found by making k as large as possible while maintaining that $kd_2 - d_1 \leq m$ (we do not have to worry about the numerator since it is less than or equal to the denominator). Hence, we should choose $k = \left\lfloor \frac{m+d_1}{d_2} \right\rfloor$, leading to, as claimed,

$$\frac{n_3}{d_3} = \frac{\left\lfloor \frac{m+d_1}{d_2} \right\rfloor n_2 - n_1}{\left\lfloor \frac{m+d_1}{d_2} \right\rfloor d_2 - d_1}.$$

□

For example, if we let $\frac{n_1}{d_1} = \frac{2}{5}, \frac{n_2}{d_2} = \frac{1}{2}$, we find that $\left\lfloor \frac{m+d_1}{d_2} \right\rfloor = \left\lfloor \frac{5+5}{2} \right\rfloor = 5$. Then, $\frac{n_3}{d_3} = \frac{5\times 1-2}{5\times 2-5} = \frac{3}{5}$, which is correct.

3 The Stern–Brocot Tree

In this section, we will define the Stern–Brocot tree, prove its key properties, and relate it to Farey sequences.

Definition 3.1. The *Stern–Brocot Tree* is an infinite binary tree consisting of rational numbers. It is rooted at a node with value $\frac{1}{1}$, and each descendant is determined as follows. Let a node s have parent of value $\frac{n_1}{d_1}$. Then,

- if s is a left-child, let $\frac{n_2}{d_2}$ be the value of the deepest ancestor of s such that s is in the right subtree of that ancestor (if there is no such node, let $\frac{n_2}{d_2} = \frac{0}{1}$).
- If s is a right-child, let $\frac{n_2}{d_2}$ be the value of the deepest ancestor of s such that s is in the left subtree of that ancestor (if there is no such node, let $\frac{n_2}{d_2} = \frac{1}{0}$).

We will assign to node s a value of $\frac{n_1+n_2}{d_1+d_2}$. Then, we will let the value of node s be denoted as v_s, the left and right children be denoted as l_s, r_s respectively, and the parent as p_s.

For example, consider computing the value of the right child of the node with value $\frac{2}{3}$. Then, node s has parent value $\frac{n_1}{d_1} = \frac{2}{3}$. The deepest ancestor value such that s is in that ancestor's left subtree is $\frac{n_2}{d_2} = \frac{1}{1}$. Thus, $v_s = \frac{2+1}{3+1} = \frac{3}{4}$. For a visualization of the tree, see Figure 2. The Stern–Brocot Tree can also be defined in terms of continued fractions.

Definition 3.2. Given a sequence $a = [a_0; a_1, a_2, \dots, a_k]$ where a_0 is a nonnegative integer and the rest are positive integers, let $\nu(a) = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$. Then, we say that a is a *continued fraction representation* of $\nu(a)$. There turns out to be a unique sequence a satisfying $\nu(a) = v$ and $a_k > 1$ (except in the degenerate case where v is an integer). In this case, we use $\mu(v)$ to denote this sequence.

Proposition 3.3. *Let $\mu(v_i) = [a_0; a_1, a_2, \dots, a_k]$. Then, $v_{p_i} = \nu([a_0; a_1, a_2, \dots, a_k - 1])$ (recall that p_i denotes the parent of i in the Stern–Brocot tree).*

Proof. We will prove this indirectly, by constructing a tree with the property described above and recovering Definition 3.1 from it. More specifically, consider the tree formed by assigning to each node i the parent node with value $\nu([a_0; a_1, a_2, \dots, a_k - 1])$ (and rearranging the children of each node to be in increasing order from left to right). Let $\mu(v_i) = [a_0; a_1, \dots, a_k]$. Then, if k is even, decreasing a_k decreases the value, while decreasing a_k increases the value for odd k. Hence, i is a right child if k is even, while i is a left child if k is odd. In either case, the value of the parent of i is $\nu([a_0; a_1, \dots, a_k - 1])$ while the value of the deepest ancestor j satisfying either $v_{p_i} < v_i < v_j$ or $v_{p_i} > v_i > v_j$ is $\nu([a_0; a_1, \dots, a_{k-1}])$. It is not difficult to show that $\nu([a_0; a_1, a_2, \dots, a_{k-1}]) = \frac{w + x a_{k-1}}{y + z a_{k-1}}$ for integers w, x, y, z depending only on $a_0, a_1, \dots, a_{k-2}$. Then,

$$\nu([a_0; a_1, a_2, \dots, a_k - 1]) = \frac{w + x\left(a_{k-1} + \frac{1}{a_k - 1}\right)}{y + z\left(a_{k-1} + \frac{1}{a_k - 1}\right)} = \frac{w(a_k - 1) + x\left(a_{k-1}(a_k - 1) + 1\right)}{y(a_k - 1) + z\left(a_{k-1}(a_k - 1) + 1\right)}.$$

We will take it for granted that this fraction does not reduce. Then, the mediant of these two values is

$$\frac{w a_k + x\left(a_{k-1} a_k + 1\right)}{y a_k + z\left(a_{k-1} a_k + 1\right)},$$

which is equal to $\nu([a_0; a_1, \dots, a_k])$ as claimed. $\square$

From Definition 3.1, we see a potential connection between the Stern–Brocot Tree and Farey sequences. Namely, Farey sequences are generated by starting with a sequence consisting of $\frac{0}{1}, \frac{1}{1}$ and repeatedly inserting the mediant of two adjacent elements into our sequence, while the Stern–Brocot tree does the same thing but in tree form. This leads to the following theorem.

Theorem 3.4. *The Farey sequence F_n can be obtained by an inorder traversal of the Stern–Brocot tree where we backtrack whenever a denominator is greater than n.*

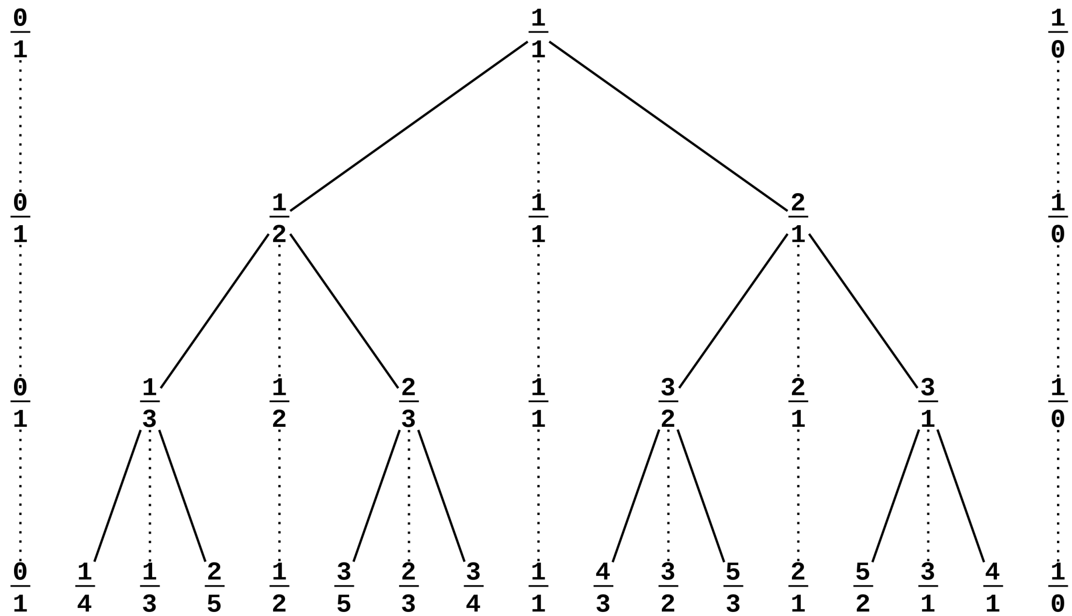

Figure 2: The Stern–Brocot Tree shown to depth 3 (taken from Wikipedia).

Proof. Let node i have value $v_i = \frac{n_i}{d_i}$ and left and right children l_i, r_i. Then, Theorem 3.4 is equivalent to saying that we can find F_n using the recursive formula

$$f_n(i) = \begin{cases} () & v_i > 1 \text{ or } d_i > n_i \\ f_n(l_i) \oplus (v_i) \oplus f_n(r_i) & \text{otherwise,} \end{cases}$$

where $\oplus$ represents the concatenation operator. Then, if we let the node with value $\frac{1}{1}$ be 1, $F_n = \left(\frac{0}{1}\right) \oplus f_n(1)$.

Since all values in the left subtree of i have values that are the mediant of fractions $\leq v_i$ and all values in the right subtree of i have values that are the mediant of fractions $\geq v_i$, $f_n(i)$ will definitely be in increasing order. To show that $F_n = (0) \oplus f_n(1)$, we need to show that all proper fractions with denominator $\leq n$ are present in our tree and that for any node i, $d_i \geq d_{p_i}$. Both of these follow as corollaries from Proposition 3.3, so we are finished. □

4 Rational Approximations

The Stern–Brocot tree, being a binary search tree, can be used to find good rational approximations to irrational numbers. Specifically, if we wish to obtain an approximation of x, we may start at $\frac{1}{1}$, and whenever the current value is greater than x, we go left, and whenever the current value is less than x, we go right. Let us examine the power of such an approximation through the following proposition.

Proposition 4.1. *For any irrational number x (where we assume $0 < x < 1$ for convenience) and any positive integer N, there exists a rational number $\frac{n}{d}$ with $0 \leq n \leq d \leq N$ such that $\left|x - \frac{n}{d}\right| < \frac{1}{Nd}$.*

Proof. Consider $\frac{n_1}{d_1}, \frac{n_2}{d_2} \in F_N$ such that $\frac{n_1}{d_1} < x < \frac{n_2}{d_2}$. Further, consider the element $\frac{n_1+n_2}{d_1+d_2}$, and, WLOG, let

$\frac{n_1}{d_1} < x < \frac{n_1+n_2}{d_1+d_2}$. Then,

$$\begin{aligned} x - \frac{n_1}{d_1} &< \frac{n_1+n_2}{d_1+d_2} - \frac{n_1}{d_1} \\ &= \frac{1}{d_1(d_1+d_2)} \\ &< \frac{1}{d_1 N}. \end{aligned}$$

□

Not only can we use this to find very good approximations of irrational numbers, but we can use it to find infinitely many good approximations of irrational numbers.

Proposition 4.2. *For any irrational number x (which we once again assume satisfies $0 < x < 1$), there are infinitely many rationals $\frac{n}{d}$ with $\left|x - \frac{n}{d}\right| < \frac{1}{d^2}$.*

Proof. Note that by Proposition 4.1, there must exist at least one such rational. Now, assume toward contradiction that there are only finitely many such rationals (which we will denote $\frac{n_1}{d_1}, \frac{n_2}{d_2}, \ldots, \frac{n_k}{d_k}$). Then, let $N > \max_i \frac{d_i}{|d_i x - n_i|}$. By Proposition 4.1, we can find a new rational, $\frac{n_{k+1}}{d_{k+1}}$, which approximates x to within $\frac{1}{d_{k+1}N} \leq \frac{1}{N}$, which is better than all previous approximations. This approximation will also yield $\left|x - \frac{n_{k+1}}{d_{k+1}}\right| < \frac{1}{d_{k+1}^2}$, and since this is closer to x than other approximations, it must be a new approximation, leading to the desired contradiction. □

However, by using the same ideas, and just working a little bit harder, we may do better.

Theorem 4.3 (Hurwitz's Theorem)**.** *For any irrational number x (which we tacitly assume satisfies $0 < x < 1$), there are infinitely many rationals $\frac{n}{d}$ satisfying $\left|x - \frac{n}{d}\right| < \frac{1}{\sqrt{5}d^2}$.*

Proof. Our proof will be based off the Farey sequence F_N (for an arbitrarily chosen N). Thus, we may simply use this Farey sequence to show the existence of such a rational, and as we scale N to be larger, we may attain infinitely many rationals. A more formal proof of a similar idea is given in the proof of Proposition 4.2.

Let $\frac{n_1}{d_1} < x < \frac{n_2}{d_2}$ for $\frac{n_1}{d_1}, \frac{n_2}{d_2} \in F_N$, and further let $\frac{n_3}{d_3} = \frac{n_1+n_2}{d_1+d_2}$. Assume WLOG that $x < \frac{n_3}{d_3}$. Then, assume towards contradiction that none of these fractions serve as suitable approximations. This gives us

$$\begin{aligned} x - \frac{n_1}{d_1} &\geq \frac{1}{\sqrt{5}d_1^2} \\ \frac{n_3}{d_3} - x &\geq \frac{1}{\sqrt{5}d_3^2} \\ \frac{n_2}{d_2} - x &\geq \frac{1}{\sqrt{5}d_2^2}. \end{aligned}$$

Algebraic manipulation gives,

$$\begin{aligned} \frac{n_2}{d_2} - \frac{n_1}{d_1} &= \frac{1}{d_1 d_2} \\ &\geq \frac{1}{\sqrt{5}}\left(\frac{1}{d_2^2} + \frac{1}{d_1^2}\right) \\ \frac{n_3}{d_3} - \frac{n_1}{d_1} &= \frac{1}{d_1 d_3} \\ &\geq \frac{1}{\sqrt{5}}\left(\frac{1}{d_3^2} + \frac{1}{d_1^2}\right). \end{aligned}$$

Clearing denominators gives us

$$\sqrt{5}d_1 d_2 \geq d_1^2 + d_2^2$$
$$\sqrt{5}d_1 d_3 = \sqrt{5}d_1(d_1 + d_2) \geq d_1^2 + d_3^2 = 2d_1^2 + d_2^2 + 2d_1 d_2.$$

Adding, we get

$$\sqrt{5}d_1(d_1 + 2d_2) \geq 3d_1^2 + 2d_2^2 + 2d_1 d_2.$$

Rearranging,

$$\begin{aligned} 0 &\geq (3 - \sqrt{5})d_1^2 + 2(1 - \sqrt{5})d_1 d_2 + 2d_2^2 \\ &= \frac{1}{2}\left((\sqrt{5} - 1)d_1 - 2d_2\right)^2. \end{aligned}$$

Thus, $(\sqrt{5} - 1)d_1 = 2d_2$ and so $\frac{\sqrt{5}-1}{2} = \frac{d_2}{d_1}$, which is impossible, because $\frac{\sqrt{5}-1}{2}$ is irrational. Thus, we have our desired contradiction. □