SPECTRAL THEOREM

TAIGA NISHIDA

Abstract. In this paper, we cover the proof of the Spectral Theorem and some applications.

1. Spectral Theorem

We introduce the Spectral Theorem.

Theorem 1.1 (Spectral Theorem)**.** *Any symmetric matrix is diagonalizable.*

We start by going over the basis and the eigenbasis.

Definition 1.2. The *basis* is like a coordinate system that relies on a set of n vectors each of size n. If for some vector $\vec{v}$, $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n}$, then $\vec{v}$ in basis $\mathcal{A} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$, represented as $[\vec{v}]_{\mathcal{A}}$ is equal to $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. For example, $\mathbb{R}^3$ uses the basis $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$, and the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in basis $\mathcal{U} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$ is equal to $\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Theorem 1.3. *For vector $\vec{x}$ and basis $\mathcal{B} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$, $\vec{x} = S[\vec{x}]_{\mathcal{B}}$ where* $S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$.

Proof. Let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. Then,

$$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n} = [\vec{v_1}, \vec{v_2}, \cdots \vec{v_n}] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = S[\vec{x}]_{\mathcal{B}}.$$

■

Definition 1.4. The eigenbasis of a matrix with eigenvectors $\vec{v_1}, \vec{v_2}, \cdots \vec{v_n}$ is the basis $\mathcal{B} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$.

Theorem 1.5. *Let there be a transformation matrix A. Let the matrix B be the $\mathcal{B}$-matrix of A, which performs the transformation of A in basis $\mathcal{B} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$ (i.e. $[A\vec{x}]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$). Then, $AS = SB$ where $S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$.*

Proof. We know that $AS[\vec{x}]_{\mathcal{B}} = A\vec{x}$. We also know that $AS[\vec{x}]_{\mathcal{B}} = S[A\vec{x}]_{\mathcal{B}} = A\vec{x}$. Therefore,

$$AS[\vec{x}]_{\mathcal{B}} = AS[\vec{x}]_{\mathcal{B}} \Longrightarrow AS = SB.$$

■

We now prove the Spectral Theorem.

Theorem 1.6. *Let there be a matrix A and eigenbasis $\mathcal{D} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$ of A with $A\vec{v_i} = \lambda_i \vec{v_i}$. Then the $\mathcal{D}$-matrix D of A is*

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \textit{ where } S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$$

Proof. We know that $D = S^{-1}AS$ is true from rearranging Theorem 1.5. We must prove that the $\mathcal{D}$-matrix of A is the diagonal matrix with the eigenvalues of A. Let there be a vector $\vec{x} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n}$. Then, $D[x]_{\mathcal{D}} = [Ax]_{\mathcal{D}}$. We see that

$$\begin{aligned} Ax &= A(c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n}) \\ &= c_1A\vec{v_1} + c_2A\vec{v_2} + \cdots c_nA\vec{v_n} \\ &= c_1\lambda_1\vec{v_1} + c_2\lambda_2\vec{v_2} + \cdots c_n\lambda_n\vec{v_n}. \end{aligned}$$

Therefore, $[Ax]_{\mathcal{D}} = \begin{bmatrix} c_1\lambda_1 \\ c_2\lambda_2 \\ \vdots \\ c_n\lambda_n \end{bmatrix}$. We know that $D[x]_{\mathcal{D}} = D\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. Thus, since $D[x]_{\mathcal{D}} = [Ax]_{\mathcal{D}}$ it is obvious that

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

■

2. Matrix Factorization Application

We present another method of factorization using Spectral Theorem.

Theorem 2.1. *A positive semidefinite real matrix has eigenvalues that are nonnegative.*

Proof. Let the matrix be A. Since it is positive semidefinite, $x^TMX \geq 0$. Let $\vec{v}$ be an eigenvector of A. Then,

$$v^TMv = v^T\lambda v = v^Tv\lambda \geq 0.$$

Since v^Tv must be nonnegative, λ, which is the eigenvalue of $\vec{v}$ must also be nonnegative. ■

This will come in use later.

Definition 2.2. A real matrix M is said to be *positive semidefinite* iff, $\forall x \in \mathbb{R}^n, x^T M x \geq 0$, where M^T is the transpose of M.

Theorem 2.3. *A real symmetric $n \times n$ matrix A is positive semidefinite if there exists an $n \times n$ real matrix X such that $A = X^T X$.*

Proof. Since $A = X^T X$, we know that $x^T A x = x^T X^T X x = (Xx)^T (Xx) = ||Xx||^2 \geq 0$.
We now prove the other direction:
Let A be a real symmetric $n \times n$ matrix. Therefore, it is positive semidefinite and by the Spectral Theorem, is diagonalizable: $D = S^{-1} A S \Longrightarrow A = SDS^{-1}$ where $S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$ and $\vec{v_1}, \vec{v_2}, \cdots \vec{v_n}$ are the eigenvectors of A and D is a diagonal matrix containing the eigenvalues of A. Let $T = S^{-1}$, so $A = T^{-1} D T$.

Theorem 2.4. *The eigenvectors of a symmetric matrix are orthogonal.*

Proof. For any matrix A and vectors $\mathbf{x}$ and $\mathbf{y}$, we know that

$$\langle A\mathbf{x}, \mathbf{y} \rangle = (A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \langle \mathbf{x}, A^T \mathbf{y} \rangle.$$

Now let A be a symmetric matrix and $\mathbf{x}$ and $\mathbf{y}$ be its eigenvectors. Let the corresponding distinct eigenvalues be λ and μ, respectively. Therefore,

$$\lambda \langle \mathbf{x}, \mathbf{y} \rangle = \langle \lambda \mathbf{x}, \mathbf{y} \rangle = \langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T \mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mu \mathbf{y} \rangle = \mu \langle \mathbf{x}, \mathbf{y} \rangle.$$

Thus, $\lambda \langle \mathbf{x}, \mathbf{y} \rangle = \mu \langle \mathbf{x}, \mathbf{y} \rangle \Longrightarrow (\lambda - \mu) \langle \mathbf{x}, \mathbf{y} \rangle = 0$. Since $\lambda \neq \mu$, then $\lambda - \mu \neq 0$ and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ implying that $\mathbf{x} \perp \mathbf{y}$. ■

Therefore, T is orthogonal and $T^{-1} = T^T$, so $A = T^T D T$.
Let $R = \sqrt{D} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\lambda_n} \end{bmatrix}$. Since all eigenvalues are nonnegative by Theorem 4.1, R remains a real matrix. Define the matrix $X = RT$. We can now see that another factorization of A is $X^T X = (RT)^T RT = T^T R^T R T = T^T D T = A$. ■

3. Matrix Power Calculation

We present a quick method of calculating matrix powers.

Let A be a real symmetric $n \times n$ matrix. The matrix's mth power is the resulting matrix after being multiplied to the identity matrix m times. For example, we have

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Then,

$$\begin{aligned}
A^3 &= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \times 1 + 2 \times 2 & 1 \times 2 + 2 \times 3 \\ 2 \times 1 + 3 \times 2 & 2 \times 2 + 3 \times 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} \\
&= \begin{pmatrix} 1 \times 5 + 2 \times 8 & 1 \times 8 + 2 \times 13 \\ 2 \times 5 + 3 \times 8 & 2 \times 8 + 3 \times 13 \end{pmatrix} \\
&= \begin{pmatrix} 21 & 34 \\ 34 & 55 \end{pmatrix}.
\end{aligned}$$

When calculating the mth power of A, it requires $n^3(m-1)$ multiplications.

Instead, by using Spectral Theorem, we can factor this into SDS^{-1} where D is the diagonal matrix containing the eigenvalues and S is the matrix with column eigenvectors. Then we have

$$\begin{aligned}
A^m &= SDS^{-1}SDS^{-1} \cdots SDS^{-1} \\
&= SD^mS^{-1} \\
&= \begin{pmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{pmatrix} \begin{pmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^m \end{pmatrix} \begin{pmatrix} \vec{v_1} \\ \vec{v_2} \\ \vdots \\ \vec{v_n} \end{pmatrix},
\end{aligned}$$

which nicely limits the number of multiplications to $n^3 + n^2 + n(m-1)$ where the power of the eigenvalues are calculated naively without optimization.

References

[1] Bretscher, Otto. *Linear algebra with applications.* Eaglewood Cliffs, NJ: Prentice Hall, 332-333, 1997.

[2] Quinlan, Rachel. *Diagonalizability of symmetric matrices.* Galway, Ireland: National University of Ireland, 2017.