LAW OF THE ITERATED LOGARITHM

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1. Introduction

The law of the iterated logarithm can be seen as a refinement of the law of large numbers and central limit theorem. Consider the number of successes in a coin-tossing game, modeled by the sum S_n of independently, identically distributed random variables $X_1, X_2, \dots, X_n$ where $X_i = +1$ with probability p and $X_i = 0$ with probability $q = 1 - p$. The mean for X_i is $\mu = p$ and the standard deviation is $\sigma^2 = p(1-p)$.

Theorem 1.1 (Strong Law of Large Numbers)**.** *The* strong law of large numbers *says that*

$$\lim_{n\to\infty} \frac{S_n - np}{n} = 0$$

with probability 1*.*

Theorem 1.2 (Central Limit Theorem)**.** *The central limit theorem applied to S_n says that*

$$\lim_{n\to\infty} \frac{S_n - np}{\sqrt{np(1-p)}} = Z$$

where Z is a random variable following the standard normal distribution $N(0,1)$.

In both theorems, we compare the limiting size of the deviation $S_n - np$ to a function of n: n for LLN and $c_1\sqrt{n}$ for CLT. LLN tells us that n grows too quickly relative to $S_n - np$ to retain any useful information about the deviation as $n \to \infty$. CLT does a better job, since $\frac{S_n - np}{c_1\sqrt{n}}$ converges to a non-trivial probability distribution. However, CLT doesn't tells us what happens for any particular sequence of coin flips, only the distribution of S_n for large n. In fact, using the Kolmogorov zero-one law and the central limit theorem, almost surely

$$\liminf_{n\to\infty} \frac{S_n - np}{\sqrt{np(1-p)}} = -\infty$$

and almost surely

$$\limsup_{n\to\infty} \frac{S_n - np}{\sqrt{np(1-p)}} = +\infty.$$

In particular, the sequence $\frac{S_n}{\sqrt{np(1-p)}}$ diverges with probability 1. We hope to find a function $f(n)$ that grows more quickly than $c_1\sqrt{n}$ but more slowly than n such that we can say something stronger about the convergence of $\frac{S_n - np}{f(n)}$.

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Theorem 1.3 (Hausdorff Estimate). *Hausdorff's estimate says that for all values of $\epsilon > 0$,*

$$\lim_{n\to\infty} \left| \frac{S_n - np}{n^{1/2+\epsilon}} \right| = 0$$

with probability 1.

Therefore, $n^{1/2}$ grows too slowly but any $n^{1/2+\epsilon}$ grows too quickly compared to the limiting deviation $S_n - np$ to perfectly capture the convergence and variation in the sequence S_n. The "right" function of n is something only slightly bigger than $\sqrt{n}$. Thus, we may try the function $\sqrt{n \log n}$, which grows slower than $n^{1/2+\epsilon}$ for any $\epsilon > 0$.

Theorem 1.4 (Hardy, Littlewood). *Hardy and Littlewood's estimate tells us that*

$$\lim_{n\to\infty} \left| \frac{S_n - np}{\sqrt{n \log n}} \right| \leq \textit{constant}$$

with probability 1. *Note that* log *is base e.*

This is a better estimate, since the information about $S_n - np$ does not condense to a single value or trail off to infinity. But still, we lose a lot of information since the limit may end up being very close to 0. Thus, $\sqrt{n \log n}$ is still a bit too strong.

Theorem 1.5 (Law of the Iterated Logarithm). *Khinchin's* law of the iterated logarithm *states that with probability* 1*,*

$$\limsup_{n\to\infty} \frac{S_n - np}{\sqrt{2np(1-p)\log\log n}} = 1$$

and symmetrically with probability 1*,*

$$\liminf_{n\to\infty} \frac{S_n - np}{\sqrt{2np(1-p)\log\log n}} = -1.$$

Now the law of the iterated logarithm tell us that $\sqrt{2np(1-p)\log\log n}$ is the "right" function to compare $S_n - np$ to. With probability 1, the ratio $\frac{S_n - np}{\sqrt{2np(1-p)\log\log n}}$ gets close to ± 1 infinitely many times, illustrating a form of recurrence. A corollary is that the random walk on $\mathbb{Z}$ hits every integer with probability 1.

2. Preliminary Definitions and Lemmas

In this section, we cover basic probability notions of limsup/limsup, state the Borel-Cantelli lemmas, and illustrate their connections to the law of the iterated logarithm.

Definition 2.1. The *limit superior* and *limit inferior* of a sequence (x_n) are refined notions of the limit, defined as follows:

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{m \geq n} x_m \right)$$

and

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{m \geq n} x_m \right).$$

Example. Let (A_n) be the sequence $0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \ldots$. Clearly, the standard limit of the sequence $\lim_{n\to\infty} A_n$ does not exist because the terms oscillate between 0 and a large positive value. However, the $\liminf$ exists and equals 0 because the infimum of the tail of the sequence is always 0. The $\limsup$ does not exist since the supremum of the tail of the sequence is always ∞.

Definition 2.2. Let $(A_n)_{n\geq 1}$ be a sequence of events in a probability space. Define the event $\{A_n\ i.o.\}$ to be where A_n occurs infinitely often in the sequence of events.

When discussing the probability $\mathbb{P}[\{A_n\ i.o.\}]$, we may drop the braces and write $\mathbb{P}[A_n\ i.o.]$ to simplify notation.

Example. Let $\omega = \{H, T\}$ be the space of coin flips where H and T come up with equal probability 0.5, and let A_n be event we get H on the nth flip. Then $\mathbb{P}[A_n\ i.o.] = 1$ since the probability that only a finite number of heads or a finite number of tails come up is 0.

The Borel-Cantelli lemmas are two important theorems that tells us when $\mathbb{P}[A_n\ i.o.] \in \{0, 1\}$.

Theorem 2.3 (First Borel-Cantelli Lemma)**.** *If*

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] < \infty,$$

then

$$\mathbb{P}[A_n\ i.o.] = 0.$$

Proof. See [Dun17, Theorem 2]. ∎

Lemma 2.4 (Second Borel-Cantelli Lemma)**.** *If $(A_n)_{n\geq 1}$ is a sequence of independent events and if*

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \infty,$$

then

$$\mathbb{P}[A_n\ i.o.] = 1.$$

Proof. See [Dun17, Theorem 3]. ∎

Example. In the previous example, with the sequence of fair coin flips, we had $\mathbb{P}[A_n] = 0.5$ for all $n \geq 1$, so the second Borel-Cantelli lemma tells us that $\mathbb{P}[A_n\ i.o] = 1$, as we predicted before. Suppose we had another sequence of coin flips but where the nth flip is with an unfair coin so that if A_n is the event that the nth flip comes up H, then $\mathbb{P}[A_n] = 2^{-n}$. Then, the first Borel-Cantelli lemma tells us that $\mathbb{P}[A_n\ i.o] = 0$ since $\sum_{n=1}^{\infty} \mathbb{P}[A_n] = 1 < \infty$.

The following equivalent formulation of the Law of the Iterated Logarithm illustrates how the Borel-Cantelli lemmas will be used in the proof of the theorem. To simplify notation, define the function

$$\alpha(n) = \sqrt{2np(1-p)\log\log n}.$$

Theorem 2.5 (Khinchin)**.** *For any $\epsilon > 0$, let A_n be the event that on the nth flip,*

$$\frac{S_n - np}{\alpha(n)} > 1 - \epsilon. \tag{2.1}$$

Then, $\mathbb{P}[A_n \text{ i.o.}] = 1$. *That is, with probability* 1*, there exist infinitely many* n *such that* (2.1) *holds. Furthermore, if* B_n *is the event such that*

$$\frac{S_n - np}{\alpha(n)} > 1 + \epsilon, \tag{2.2}$$

then $\mathbb{P}[B_n \text{ i.o.}] = 0$. *That is, with probability* 1*, there will only be finitely many* n *such that* (2.2) *holds.*

The two conditions are equivalent to a lower bound by $1-\epsilon$ and an upper bound by $1+\epsilon$ on the lim sup in Theorem 1.5, respectively.

3. Proving the Law of the Iterated Logarithm

We first state and prove some helpful lemmas. The first lemma gives an upper and lower bound on the probability that S_n deviates a significant amount from the mean np, relative to $\alpha(n)$.

Lemma 3.1. *For all positive* a *and* δ *and large enough* n*,*

$$(\log n)^{-a^2(1+\delta)} < \mathbb{P}[S_n - np > a\alpha(n)] < (\log n)^{-a^2(1-\delta)}.$$

Proof. The proof uses the Large and Moderate Deviations theorems, which is too large of a digression for this paper, so we omit it. For the proof, see [Dun18, Lemma 5]. ■

The next theorem relates the maximum deviation (or fluctuation) from step 1 to n to the deviation at step n. Thus, to get decent bounds for all steps from 1 to n, we just have to get good enough bounds on the nth step.

Lemma 3.2 (Kolmogorov Maximal Inequality)**.** *Let* $(Y_n)_{n\geq 1}$ *be a sequence of independent random variables with* $\mathbb{E}[Y_n] = 0$ *and* $\operatorname{Var}(Y_n) = \sigma^2$. *Define* $T_n = Y_1 + \cdots + Y_n$. *Then,*

$$\mathbb{P}\left[\max_{1\leq k\leq n} T_k \geq b\right] \leq \frac{4}{3}\mathbb{P}\left[T_n \geq b - 2\sigma\sqrt{n}\right].$$

Remark 3.3. Lemma 3.2 is an example of a class of lemmas called *maximal inequalities*. An example of a maximal equality from the Euler Circle Markov chains class is Problem 7 of Week 5: if X_t is the random walk on $\mathbb{Z}$, then

$$\mathbb{P}\left[\max_{1\leq k\leq n} X_k \geq b\right] \leq 2\mathbb{P}[X_n \geq b].$$

The proof of Lemma 3.2 is not hard if we assume Chebyshev's inequality, which states a general relationship between variance and the deviation $|X-\mu|$:

Lemma 3.4 (Chebyshev's Inequality)**.** *Let* X *be a random variable with finite mean* μ *and finite, non-zero variance* σ^2. *Then for all* $c > 0$*,*

$$\mathbb{P}[|X-\mu| \geq c\sigma] \leq \frac{1}{c^2}.$$

Proof of Lemma 3.2. Since the Y_k's are independent,

$$\operatorname{Var}(T_n - T_k) = \operatorname{Var}(Y_{k+1} + Y_{k+2} + \cdots + Y_n) = (n-k)\operatorname{Var}(Y_1) = (n-k)\sigma^2$$

for all $1 \leq k \leq n$. Using Lemma 3.4 with $X = T_n - T_k$ and $c = \frac{2\sqrt{n}}{\sqrt{n-k}}$ gives

$$\mathbb{P}[|T_n - T_k| \leq 2\sigma\sqrt{n}] \geq 1 - \frac{n-k}{4n} \geq \frac{3}{4}.$$

Note that

$$\begin{aligned}
\mathbb{P}\left[\max_{0\le k\le n} T_k \ge b\right] &= \sum_{k=1}^{n} \mathbb{P}[T_1 < b, \dots, T_{k-1} < b, T_k \ge b] \\
&\le \sum_{k=1}^{n} \mathbb{P}[T_1 < b, \dots, T_{k-1} < b, T_k \ge b] \cdot \frac{4}{3}\mathbb{P}[|T_n - T_k| \le 2\sigma\sqrt{n}] \\
&= \frac{4}{3}\sum_{k=1}^{n} \mathbb{P}[T_1 < b, \dots, T_{k-1} < b, T_k \ge b, |T_n - T_k| \le 2\sigma\sqrt{n}] \\
&\le \frac{4}{3}\sum_{k=1}^{n} \mathbb{P}[T_1 < b, \dots, T_{k-1} < b, T_k \ge b, T_n \ge b - 2\sigma\sqrt{n}] \\
&\le \frac{4}{3}\mathbb{P}[T_n \ge b - 2\sigma\sqrt{n}].
\end{aligned}$$

■

3.1. **Pseudo-proof of Theorem 1.5.** In this subsection, we give simplified proof that partially proves and motivates the actual proof of Theorem 1.5. Recall the two conditions (2.1) and (2.2) equivalent to the theorem. Proving the lower bound (2.1) only requires finding an appropriate subsequence with a sufficiently large limit, but in fact, this is hard because second Borel-Cantelli lemma requires independence.

We may partially address the upper bound (2.2) with the following argument. Fix some $\gamma > 1$ and let $n_k = \lfloor \gamma^k \rfloor$. Lemma 3.1 tells us that for any positive δ,

$$\begin{aligned}
\mathbb{P}[S_{n_k} - pn_k \ge (1+\epsilon)\alpha(n_k)] &< (\log n_k)^{-(1+\epsilon)^2(1-\delta)} \\
&= O\left(k^{-(1+\epsilon)^2(1-\delta)}\right)
\end{aligned}$$

for sufficiently large n. The big-O notation means that the probability grows less than a fixed multiple of the inside function. Choose δ so that $c = -(1+\epsilon)^2(1-\delta) < -1$. This gives

$$\sum_{k=1}^{\infty} \mathbb{P}[S_{n_k} - pn_k \ge (1+\epsilon)\alpha(n_k)] < \sum_{k=1}^{\infty} O(k^c) < \infty.$$

Now, the first Borel-Cantelli lemma tells us that

$$\mathbb{P}[S_{n_k} - pn_k \ge (1+\epsilon)\alpha(n_k)\ i.o.] = 0,$$

so

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_{n_k} - pn_k}{\alpha(n_k)} < 1+\epsilon\right] = 1.$$

3.2. **Proof of Theorem 1.5.** In this subsection, we will fully prove Theorem 1.5. First, let's show that the result from the previous section holds if we replace $\{n_k\}$ with $\{n \ge 1\}$.

Proof of (2.1). Fix $\epsilon > 0$ and let $\gamma > 1$ be a constant chosen later. Like before, let $n_k = \lfloor \gamma^k \rfloor$. Our goal is to show that

$$\sum_{k=1}^{\infty} \mathbb{P}\left[\max_{n\le n_{k+1}} (S_n - np) \ge (1+\epsilon)\alpha(n_k)\right] < \infty.$$

To simplify notation, let $R_n = S_n - np$ be the deviation. From Lemma 3.2,

$$\mathbb{P}\left[\max_{n\le n_{k+1}} R_n \ge (1+\epsilon)\alpha(n_k)\right] \le \frac{4}{3}\mathbb{P}\left[R_{n_{k+1}} \ge (1+\epsilon)\alpha(n_k) - 2\sqrt{n_{k+1}p(1-p)}\right]. \tag{3.1}$$

Note that $\sqrt{n_{k+1}} = o(\alpha(n_k))$ since

$$\sqrt{n_{k+1}} \sim \sqrt{\gamma^{k+1}}$$

and

$$\alpha(n_k) = \sqrt{2p(1-p)n_k \log\log n_k} \sim c_1\gamma^{k/2}\sqrt{\log k + \log\log\gamma}.$$

Dividing both terms by $\gamma^{k/2}$, we see that $\gamma^{1/2}$ is constant while $c_1\sqrt{\log k + \log\log\gamma}$ goes to infinity. Thus, we conclude that $\frac{\sqrt{n_{k+1}}}{\alpha(n_k)} \to 0$. The limit remains 0 when each term is multiplied by a non-zero constant, so $2\sqrt{n_{k+1}p(1-p)} < \frac{1}{2}\epsilon\alpha(n_k)$ for sufficiently large n. Using this inequality on the right side of (3.1) gives

$$\begin{aligned}\mathbb{P}\left[\max_{n\le n_{k+1}} R_n \ge (1+\epsilon)\alpha(n_k)\right] &\le \frac{4}{3}\mathbb{P}\left[R_{n_{k+1}} \ge (1+\epsilon)\alpha(n_k) - \frac{1}{2}\epsilon\alpha(n_k)\right] \\ &= \frac{4}{3}\mathbb{P}\left[R_{n_{k+1}} \ge (1+\epsilon/2)\alpha(n_k)\right].\end{aligned}$$

To turn $\alpha(n_k)$ into $\alpha(n_{k+1})$, note that $\alpha(n_{k+1}) \sim \alpha(n_k)$ i.e., their ratio goes to 1 as $k \to \infty$. Choose γ so that $1 + \epsilon/2 > (1+\epsilon/4)\sqrt{\gamma}$. Then for large enough k,

$$(1+\epsilon/2)\alpha(n_k) > (1+\epsilon/4)\alpha(n_{k+1}).$$

Now, we're almost done. Using Lemma 3.1 with $a = (1-\delta)^{-1} = (1+\epsilon/4)$ gives

$$\mathbb{P}\left[\max_{n\le n_{k+1}} R_n \ge (1+\epsilon)\alpha(n_k)\right] \le \frac{4}{3}(\log n_{k+1})^{-(1+\epsilon/4)}$$

for all large k. The right side approximates as follows:

$$(\log n_{k+1})^{-(1+\epsilon/4)} \sim (\log\gamma)^{-(1+\epsilon/4)}k^{-(1+\epsilon/4)}.$$

Since these terms converge when summed over $k \ge 1$, we have

$$\sum_{k=1}^{\infty}\mathbb{P}\left[\max_{n\le n_{k+1}}(S_n - np) \ge (1+\epsilon)\alpha(n_k)\right] < \infty$$

as desired.

To finish the proof, we use the Borel-Cantelli lemma to get

$$\max_{n\le n_{k+1}} R_n \ge (1+\epsilon)\alpha(n_k) \text{ i.o. with probability 0,}$$

or equivalently

$$\max_{n\le n_{k+1}} R_n < (1+\epsilon)\alpha(n_k) \text{ for all large } k \text{ with probability 1.}$$

In particular,

$$\max_{n_k\le n<n_{k+1}} R_n < (1+\epsilon)\alpha(n_k) \text{ for all large } k \text{ with probability 1.}$$

Since $(1+\epsilon)\alpha(n_k) \le (1+\epsilon)\alpha(n)$, we find that with probability 1, there exists some n_0 such that for all $n > n_0$,

$$R_n = S_n - np < (1+\epsilon)\alpha(n),$$

which proves that

$$\limsup_{n\to\infty} \frac{S_n - np}{\alpha(n)} < 1+\epsilon.$$

■

Proof of (2.2). It suffices to find a set $\{n_k\}$ so that with probability 1, $R_{n_k} \geq (1-\epsilon)\alpha(n_k)$ infinitely often. Let $n_k = \gamma^k$ for some sufficiently large $\gamma \in \mathbb{Z}$ chosen later. The proof will show

$$\sum_{k=1}^{\infty} \mathbb{P}\left[R_{\gamma^k} - R_{\gamma^{k-1}} \geq \left(1-\frac{\epsilon}{2}\right)\alpha(\gamma^n)\right] = \infty \tag{3.2}$$

and

$$R_{\gamma^{k-1}} \geq \frac{-\epsilon}{2}\alpha(\gamma^k) \text{ for all large enough } k\text{, with probability 1.} \tag{3.3}$$

Note that since R_n is a sum of independent random variables, $R_{\gamma^k} - R_{\gamma^{k-1}}$ has the same probability distribution as $R_{\gamma^k - \gamma^{k-1}}$. Thus, it suffices to consider

$$\mathbb{P}\left[R_{\gamma^k-\gamma^{k-1}} \geq \left(1-\frac{\epsilon}{2}\right)\alpha(\gamma^k)\right].$$

Note that

$$\begin{aligned}
\frac{\alpha(\gamma^k-\gamma^{k-1})}{\alpha(\gamma^k)} &= \sqrt{\frac{\gamma^k-\gamma^{k-1}}{\gamma^n}\frac{\log(\log(\gamma^k-\gamma^{k-1}))}{\log(\log(\gamma^k))}} \\
&= \sqrt{\left(1-\frac{1}{\gamma}\right)\frac{\log\left(k\log\gamma + \log\left(1-\frac{1}{\gamma}\right)\right)}{\log(k\log\gamma)}} \\
&\to \sqrt{1-\frac{1}{\gamma}}.
\end{aligned}$$

Choose γ so that

$$\frac{1-\frac{\epsilon}{2}}{1-\frac{\epsilon}{4}} < \sqrt{1-\frac{1}{\gamma}}.$$

Then for all large enough k,

$$\frac{1-\frac{\epsilon}{2}}{1-\frac{\epsilon}{4}} < \frac{\alpha(\gamma^k-\gamma^{k-1})}{\alpha(\gamma^k)},$$

or equivalently

$$\left(1-\frac{\epsilon}{2}\right)\alpha(\gamma^k) < \left(1-\frac{\epsilon}{4}\right)\alpha(\gamma^k-\gamma^{k-1}).$$

This gives the inequality

$$\mathbb{P}\left[R_{\gamma^k} - R_{\gamma^{k-1}} \geq \left(1-\frac{\epsilon}{2}\right)\alpha(\gamma^k)\right] \geq \mathbb{P}\left[R_{\gamma^k-\gamma^{k-1}} \geq \left(1-\frac{\epsilon}{4}\right)\alpha(\gamma^k-\gamma^{k-1})\right].$$

Now we may use Lemma 3.1 with $a = (1+\delta)^{-1} = (1-\frac{\epsilon}{4})$ to get

$$\begin{aligned}
\mathbb{P}\left[R_{\gamma^k} - R_{\gamma^{k-1}} \geq \left(1-\frac{\epsilon}{2}\right)\alpha(\gamma^k)\right] &\geq \log(\gamma^k-\gamma^{k-1})^{\left(1-\frac{\epsilon}{4}\right)} \\
&= \left(k\log\gamma + \log\left(1-\frac{1}{\gamma}\right)\right)^{-\left(1-\frac{\epsilon}{4}\right)}
\end{aligned}$$

The sum of these terms over all $k \geq 1$ diverge, thereby proving (3.2).

It isn't hard to show that $\alpha(\gamma^k) \sim \sqrt{\gamma}\alpha(\gamma^{k-1})$; we leave this as an exercise to the reader. Choose γ so that $\epsilon\sqrt{\gamma} > 4$, then $\frac{\epsilon}{2}\alpha(\gamma^k) \sim \frac{\epsilon}{2}\sqrt{\gamma}\alpha(\gamma^{k-1}) > 2\alpha(\gamma^{k-1})$ for large enough k. Thus, for large enough k,

$$\left[R_{\gamma^{k-1}} > \frac{-\epsilon}{2}\alpha(\gamma^k)\right] \supseteq \left[R_{\gamma^{k-1}} > -2\alpha(\gamma^{k-1})\right].$$

By the first part of the law of the iterated logarithm i.e., (2.1), we see that $R_{\gamma^{k-1}} > -2\alpha(\gamma^{k-1})$ for all large enough k, with probability 1. Thus, the event $R_{\gamma^{k-1}} > \frac{-\epsilon}{2}\alpha(\gamma^k)$ occurs for all large k with probability 1, thus proving (3.3).

Now since $R_{\gamma^k} - R_{\gamma^{k-1}}$ is a sequence of independent random variables, the second Borel-Cantelli lemma on (3.2) says that almost surely

$$R_{\gamma^k} - R_{\gamma^{k-1}} \geq \left(1 - \frac{\epsilon}{2}\right)\alpha(\gamma^k) \text{ i.o.}$$

Combining this with (3.3) we get that

$$R_{\gamma^k} > (1-\epsilon)\alpha(\gamma^k) \text{ i.o.}$$

almost surely. This is enough to prove that

$$\limsup_{n\to\infty} \frac{S_n - np}{\alpha(n)} \geq 1 - \epsilon$$

almost surely, finishing the proof of the Law of the Iterated Logarithm. ■

4. Further Generalizations

One obvious extension of Theorem 1.5 is letting (X_n) be a sequence of i.i.d. random variables with $\mu = \mathbb{E}[X_n]$ and $\sigma^2 = \mathrm{Var}[X_n]$. The same proof shows that almost surely,

$$\limsup_{n\to\infty} \frac{S_n - n\mu}{\sqrt{2n\sigma^2 \log\log n}} = 1$$

where $S_n = \sum_{i\leq n} X_i$.

In 1929, Kolmogorov proved a version of the LIL for independent, but not necessarily identically distributed X_n with $\mathbb{E}[X_n] = 0$. With notation as $S_n = \sum_{i\leq n} X_i$ and $s_n^2 = \mathrm{Var}[S_n]$, he assumed that $s_n^2 \to \infty$ and

$$|X_n| \leq \frac{\epsilon_n s_n}{\sqrt{\log(\log(s_n^2))}}$$

for some sequence of constants $\epsilon_n \to 0$. He showed that almost surely,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log(\log(s_n^2))}}.$$

Exercise. Show that Kolmogorov's LIL reduces to Khinchin's LIL when X_n are i.i.d.

In 1964, Strassen [Str64] obtained a more precise version of Khinchin's LIL:

Theorem 4.1. *Let S_n be the sum of the first n of a sequence of i.i.d. random variables having mean 0 and variance 1. Let $0 \leq c \leq 1$ and*

$$c_i = \begin{cases} 1 & \text{if } S_i > c\sqrt{2i\log\log i} \\ 0 & \text{otherwise} \end{cases}.$$

Then,

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{1}{n}\sum_{i=3}^{n} c_i = 1 - \exp\left\{-4\left(\frac{1}{c^2}-1\right)\right\}\right] = 1.$$

Example. For $c = \frac{1}{2}$, $1 - \exp\left\{-4\left(\frac{1}{c^2}-1\right)\right\} \approx 0.99999385$, so we obtain the surprising result that almost surely, for infinitely many of n the percentage of times $i \le n$ when

$$S_i > \frac{1}{2}\sqrt{2i \log\log i}$$

exceeds 99.999, but only for finitely many n exceeds 99.9999.

Exercise. Deduce Khinchin's LIL from Strassen's generalization.

Mathematicians have also considered LIL for Brownian motion, a continuous version of the random walk. The following result is analogous to Theorem 1.5 for Brownian motion.

Theorem 4.2. *Let $(B(t))_{t\in\mathbb{R}^+}$ be a real-valued Brownian motion with continuous sample paths. Then,*

$$\mathbb{P}\left[\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t\log\log t}} = 1\right] = 1.$$

In fact, a slightly stronger result is that

Theorem 4.3. *Let $(B(t))_{t\in\mathbb{R}^+}$ be a real-valued Brownian motion with continuous sample paths. The set of cluster points of the family of random variables*

$$\frac{B(t)}{\sqrt{2t\log\log t}} \text{ as } t \to \infty$$

is almost surely $[-1, +1]$.

In other words, with probability 1, for any $x \in [-1, +1]$, there is an infinite subsequence of $\frac{B(t)}{\sqrt{2t\log\log t}}$ with the t-values tending to infinity such that the subsequence converges to x.

For generalizations of LIL on Brownian motion in finite and infinite dimensional vector spaces and differentiable manifolds, see [Dun75]. For LIL in other contexts, see [Bin86].

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