Brownian Motion

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1 Introduction

In a certain space, such as a fluid, there are many different particles that move around randomly, such as the fluid molecules. This "randomness" is known as Brownian motion, and can be modeled using certain equations and definitions, and then those definitions can be used as applications in other models to understand certain concepts.

Random walks on a number line often start from 0, and the current position is incremented by +1 or -1. If we define $X_1, X_2, X_3...$ a state space of $\{-1, 1\}$, then by summing up the X_t values, we get the value for the random walk S_t. A similar approach can be taken to Brownian motion, although there are a few more rules:

Definition 1.1. *Brownian motion B_t is a type of motion defined to have the following characteristics:*

1. $B_0 = 0$
2. *If $0 < p < q$, then $B_q - B_p$ follows a normal distribution with mean 0 and variance $q - p$.*
3. *If $0 \leq a \leq b \leq c \leq d$, then because [a, b] and [c, d] do not overlap, $B_b - B_a$ and $B_d - B_c$ are independent variables.*
4. *$t \to B_t$ is continuous.*

2 Properties

There are certain properties that Brownian motion has, such as being stationary, continuous, and being a Gaussian process (Ermogenous).

Definition 2.1. *A stationary process means that even after any shifts in time, the mean and variance of the process stays the same. Another way of saying this is that the distribution of ($B_0,\ B_1, \ldots B_k$) has the same distribution as ($B_t,\ B_{t+1} \ldots B_{t+k}$).*

Theorem 2.1. *Brownian motion is a stationary process.*

We can see that for $1 \leq n \leq m$,

$$B_m - B_n = \sum_{i=n+1}^{m} X_i.$$

However, just by changing the increment by subtracting the upper and lower bound by n, we get the sum $B_{m-n} - B_0 = B_{m-n}$. Because the values of X_i are i.i.d., the distribution of $(B_n \ldots B_m)$ behaves the same way as the distribution $(B_0 \ldots B_{m-n})$. Thus, B_t is a stationary process.

Definition 2.2. *A Gaussian process in a type of process such that every finite collection of those random variables follows a multivariate normal distribution. This means that for variables $X_1,\ X_2 \ldots X_k$ in a Gaussian process X_n, for any constants a_n, $a_1X_1 + a_2X_2 + \ldots + a_kX_k$ will follow a normal distribution curve.*

Theorem 2.2. *Brownian motion is a Gaussian process.*

In order for B_t to be a Gaussian process, $a_1B_1 + a_2B_2 + \ldots + a_kB_k$ must follow a normal distribution. However, knowing that $B_n = \sum_i^n X_i$ makes this easy to see. The sum can be rewritten as $a_1(X_1) + a_2(X_1 + X_2) + \ldots + a_k(X_1 + X_2 + \ldots + X_k)$. Rearranging the terms, we see that the expression becomes $(a_1 + a_2 + \ldots + a_k)(X_1) + (a_2 + a_3 + \ldots + a_k)(X_2) + \ldots + a_kX_k$. Each of the X_i variables follows a normal distribution, and even after multiplying them by a constant, they still follow that distribution. In addition, adding two normally distributed variables results in a normally distributed variable again, but with added means and variances. Therefore, the sum results in a Gaussian distribution, so B_n is a Gaussian process.

Proposition 2.2.1. *The expected value of Brownian motion at time n is 0, or $\mathbb{E}[B_n] = 0$.*

We know that

$$\mathbb{E}[B_n] = \mathbb{E}[\sum_{i=1}^{n} X_i]$$

$$= \sum_{i=1}^{n} \mathbb{E}[X_i].$$

X_i behaves the same way as N(0, 1), so it can be treated as the function e^{-x^2}. To find the expected value of e^{-x^2}, we need to calculate

$$\int_{-\infty}^{\infty} xe^{-x^2} dx$$

$$= \int_{-\infty}^{\infty} odd \cdot even\ dx$$

$$= 0.$$

Therefore, $\mathbb{E}[X_i] = 0$. Incorporating this into our sum from before, we can see that

$$\sum_{i=1}^{n} \mathbb{E}[X_i]$$

$$= \sum_{i=1}^{n} 0$$

$$= 0.$$

Proposition 2.2.2. *The variance of Brownian motion at time n is n, or $Var(B_n) = n$.*

B_n can be rewritten as the sum $\sum_{i=1}^{n} X_i$, and because the sum of two normal distributions results in their variances adding, know that adding the variance 1 n times results in a variance of n, that is the variance of B_n.

Proposition 2.2.3. *The covariance between Brownian motion at times s and t is the minimum between the two times, or $Cov(B_s, B_t) = min(s, t)$.*

To prove that $\mathrm{Cov}(B_t, B_s) = \min(\text{t, s})$, we must first assume $0 \leq s \leq t$ WLOG. Furthermore, using the definitions before stating that $B_0 = 0$, and the values of $X_t = B_t - B_{t-1}$ are i.i.d., we see that:

$$\mathrm{Cov}(B_t, B_s) = \mathbb{E}[B_t \cdot B_s]$$

$$= \mathbb{E}[(B_s + B_t - B_s)B_s]$$
$$= \mathbb{E}[B_s^2 + (B_t - B_s)B_s]$$
$$= \mathbb{E}[B_s^2] + \mathbb{E}[(B_t - B_s)B_s]$$
$$= s + \mathbb{E}[(B_t - B_s)(B_s)]$$
$$= s + \mathbb{E}[B_t - B_s] \cdot \mathbb{E}[B_s]^1$$
$$= s + \mathbb{E}[B_t - B_s](0)$$
$$= s + 0$$
$$= s.$$

Because the increments follow a normal distribution that is symmetric along the y-axis, it results in some interesting properties.

Theorem 2.3. *If we define a first passage time for a as $T_a := inf\{t : B_t = a\}$, then $\mathbb{P}(B_t > a) = \frac{1}{2}\mathbb{P}(T_a < t)$.*

First of all, we notice that

$$\mathbb{P}(B_t > a) = \mathbb{P}(T_a < t, B_t > a).$$

This is because B_t is continuous, so if $B_t > a$, then we have already passed T_a, as it is defined to be the first hitting time. The second inequality is

$$\mathbb{P}(T_a < t, B_t > a) \;=\; \frac{1}{2}\mathbb{P}(T_a < t).$$

This is because $B_{T_a} = a$, so at time $t > T_a$, we know that $\mathbb{P}(B_t > a) = \mathbb{P}(B_t < a)$, seeing as B_{t-T_a} is normally distributed with its mean at 0. Furthermore, $B_t = a$ has probability 0 as the increments of t get smaller.

[1] We can do this because $B_t - B_s$ and B_s are independent, as we have proven before.

Theorem 2.4. *Brownian motion exists.*

We can start by constructing B_n on the dyadic rationals on [0,1], and then extending the process to infinity. Firstly, let

$$D_n = \{a2^{-n} : 0 \leq a \leq 2^n\},$$

and then let

$$D = \bigcup_{n=0}^{\infty} D_n.$$

Next, we can define $\{Z_t\}_{t \in D}$ to be a set of of i.i.d. normally distributed variables, with Z_i behaving similarly to N(0,1). Knowing that $B_0 = 0$ and $B_1 = Z_1$, we can construct the rest of the B_ns recursively. Because $B_1 - B_0$ is Gaussian with a variance of 1, then for $d \in D_n \backslash D_{n-1}$, we can define

$$B_d = \frac{B_{d-2^{-n}} + B_{d+2^{-n}}}{2} + \frac{Z_d}{2^{(n+1)/2}}.$$

What this is doing is finding the average of the two dyadic rationals that are closest to it in the set D_{n-1}, averaging them, and then slightly offsetting the result with a random normally distributed variable. The variable B_d does not depend on D_{n+k} for $k \geq 1$.
By looking at the neighboring increments, calculation shows that

- $B_d - B_{d-2^{-n}} = \frac{B_{d+2^{-n}} - B_{d-2^{-n}}}{2} + \frac{Z_d}{2 \cdot 2^{(n-1)/2}}$
  and
- $B_d - B_{d-2^{-n}} = \frac{B_{d+2^{-n}} - B_{d-2^{-n}}}{2} - \frac{Z_d}{2 \cdot 2^{(n-1)/2}}$.

These intervals each are Gaussian, and their variance is 2^{1-n}.We can define a specific function $F_n(t)$, where

$$F_0(t) = \begin{cases} Z_1, & t = 1 \\ 0, & t = 0 \\ \text{linearly}, & \text{all other cases} \end{cases}$$

and

$$F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t, & t \in D_n \backslash D_{n-1} \\ 0, & t \in D_n \\ \text{linearly}, & \text{all other cases} \end{cases}.$$

By calculation, we can see that

$$B_d = \sum_{i=0}^{\infty} F_i(d).$$

However, we must show that this infinite series is uniformly convergent on [0, 1]. First of all, using calculus, we can see that

$$\mathbb{P}[|Z_d| \geq c\sqrt{n}] \leq e^{-c^2 n/2}.$$

It follows to show that

$$\sum_{n=0}^{\infty} \mathbb{P}[\exists d \in D_n, |Z_d| \geq c\sqrt{n}] \leq \sum_{n=0}^{\infty} (2^n + 1) e^{-c^2 n/2} < \infty$$

as well. In fact, for a random, large N, we have $n > N$, and $|Z_d| < c\sqrt{n}$ for all $d \in D_n$. Finally, by multiplying $2^{-(n+1)/2}$ on both sides, we can see that for most values of F,

$$||F_n||_\infty < c\sqrt{n} 2^{-(n+1)/2}.$$

We know that D is dense in [0,1] on $\mathbb{R}$, so the continuity is finished. Finally, the process can be extended to all of the nonnegative real numbers by concatenating individual copies of B_n.

3 Applications

Brownian motion has certain applications in real life, and can be used for modeling in certain problems and scenarios. For example, it can be used for modeling terrain. Mobile robots have been made which have predicted and calculated possible shifts in the terrain it travels on according to random Brownian motion. It was also able to make much bigger maps by splicing together the smaller maps it had constructed, much like how the random motion was extended to the whole number line in the proof of its existence.

In addition, Brownian motion behaves like a fractal. A fractal, or fractional dimension, is a mathematical or visual object which displays certain properties. For example, looking at a random two-dimensional motion, it is self similar. This means that by "zooming in" on Brownian motion, the smaller interval will look similar to the larger motion, or the path in travels on in general.

These sorts of fractals can be found in nature, and can also help with understanding and predicting processes that behave in a similar way. For example, patterns in medical images are similar to Brownian motion, as well as changes and trends in the stock market. Therefore, because the pattern is alike, certain properties of the trends might be akin to the properties of Brownian motion. However, this does NOT mean that one should run a Brownian simulation in order to predict how to spend their money, or to study medical phenomena, as those results could be catastrophic.