THE JORDAN MEASURE & RIEMANN INTEGRAL

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1. Introduction

Measure theory is a branch of mathematics that studies the concept of a "measure." A measure, informally, serves as a generalization of lengths, areas, and volumes. Measure theory was laid out by Émile Borel, Henri Lebesgue, Hans Rademacher, and many more. Developed during the late 1800s and early 1900s, it contains numerous applications in probability theory, real analysis, topology, and much more. So, we will begin with the fundamentals.

2. Elementary and Jordan Measures

Intuitively speaking, a measure is the "size" of a given set. For instance, the measure of a subset $I \subset \mathbb{R}^1$ would be the length of I and the measure of a subset $I \subset \mathbb{R}^2$ would be the area of I. Before we head into the rigorous definitions of the elementary and Jordan measures, we begin with some axioms.

Suppose $m(A)$ is the measure of a set A. Then, the following axioms are satisfied:

(1) $m(A) = m(A + b) = m(AC)$, where b is a constant and C is a rotation matrix. In other words, the measure of a set A is preserved under translations and rotations.
(2) $m(A \sqcup B) = m(A) + m(B)$. In other words, if the set A can be broken up into disjoint pieces, then the sum of the measures of these pieces will be A.

Now, some basic definitions.

Definition 2.1. We define an *interval* as a subset of $\mathbb{R}$ which takes the form $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$, $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$, $(a, b) := \{x \in \mathbb{R} : a < x < b\}$, with $a, b \in \mathbb{R}$.

Definition 2.2. A *box* in $\mathbb{R}^d$ is a Cartesian product $B : I_1 \times ... \times I_d$ of d intervals $I_1, ..., I_d$.

Definition 2.3. Suppose we have a set X with power set S. Then, $A \subseteq S$ is a $\sigma-algebra$ if:

a) $\emptyset, X \in A$
b) If $B \in A$, then $B^c \in A$
c) For $i \in \mathbb{N}$ and $B_i \in A$, $\bigcup_{i=1}^{\infty} B_i \in A$.

Any element in a $\sigma-$algebra is called a *measurable set*.

The elements of a $\sigma-$algebra are measurable. Let's think about these conditions intuitively. Clearly, the empty set and the set X itself are going to be measurable (these are the easiest cases). For the next condition, it's nice to think about the set X as some 2-dimensional figure. An element B will naturally "take up" some of this space, and since this space is measurable, it's natural that B^c is measurable as well because this is simply the difference between space of X and B. For the final condition, we can think about each individual measurable set as comprising our A_i for $i \in \mathbb{N}$.

The $\sigma-$algebra's usefulness will become apparent when we discuss measurable spaces. For now, we have enough to define an elementary set.

Definition 2.4. An *elementary set* is any subset of $\mathbb{R}^d$ that's the union of a finite number of boxes.

The elementary measure, naturally, is the notion that allows us to measure these sets. This leads into the following lemma:

Lemma 2.5. Suppose $E \subset \mathbb{R}^d$ is an elementary set. Then:

a) E can be expressed as the finite union of disjoint boxes
b) If E is partitioned as the finite union $B_1 \cup ... \cup B_k$ of disjoint boxes, then the quantity $m(E) := |B_1|+...+|B_k|$ is independent of the partition. That is, given any other partition $C_1 \cup ... \cup C_i$ in E, one has $|B_1| + ... + |B_k| = |C_1| + ... + |C_i|$.

Proof. We begin with the proof of part a). First, let's show the case where $d = 1$. We have a finite amount of intervals $I_1, ..., I_m$, and we can place the $2m$ endpoints of these in increasing order. There also exists a finite amount of disjoint intervals $J_1, ..., J_n$ arranged such that each $I_1, ..., I_m$ is the union of some subcollection of $J_1, ..., J_n$. This suffices to prove the one-dimensional case. To prove the case of arbitrary dimension, we express E as the union of boxes $B_i = I_{i,1}\times, ..., \times I_{i,d}$ for $i = 1, ..., m$. We can also express the intervals $I_{1,k}, ..., I_{m,j}$ as the union of subcollections of $J_{1,j}, ..., J_{n_j,j}$ (these are disjoint intervals). This is true because we can apply our argument of the one-dimensional case.

Now, we can express $B_1, ..., B_k$ as finite unions of $J_{i_1,1}, ..., J_{i_d,d}$, where $1 \le i_j \le n_j$ for all $1 \le j \le d$. These boxes are disjoint, proving this claim.

For part b), we note that the length of any interval I can be found through:

$$|I| = \lim_{N\to\infty} \frac{1}{N}\#(I \cap \frac{1}{N}\mathbb{Z}).$$

Now, we can say that

$$|B| = \lim_{N\to\infty} \frac{1}{N^d}\#(B \cap \frac{1}{N}\mathbb{Z}^d),$$

where B is an arbitrary box. Since, $|B|$ is just the sum of each disjoint subset of B, the following statement follows:

$$|B_1|+,...,+|B_n| = \lim_{N\to\infty} \frac{1}{N^d}\#(E \cap \frac{1}{N^d}\mathbb{Z}^d).$$

We have that the RHS is simply $m(E)$, thus completing the proof of this lemma. ■

This lemma makes it clear that for disjoint elementary sets $E_1, ..., E_k$ for some $k \in \mathbb{N}$, $m(E_1\cup, ..., \cup E_k) = m(E_1) + ... + m(E_k)$. It also shows our translation axiom.

Now, let's move on to Jordan measures. Jordan measures, in general, help with finding an appropriate notion of the "size," or higher-dimensional analogue of volume, of subsets of $\mathbb{R}^n$. First, suppose we have a bounded subset $E \subset \mathbb{R}^n$. Let's construct a covering of an interval I from finitely many intervals such that no two contain common interior points. Now, we keep dividing this interval I in half, and those two pieces in half again, and so forth. From all these intervals, some may contain points that are on the closure of E and $\bar{E}$. We define these intervals as $E_1, ..., E_n$. We define the intervals $\tilde{E_1}, ..., \tilde{E_m}$ as the intervals containing elements that are *only* on the interior of E. Now, we can say the following:

$$S = \sum_{i=1}^{n} m(E_i),\ \tilde{S} = \sum_{i=1}^{m} m(\tilde{E_i}).$$

We define the Jordan inner measure, $m_{*,(J)}(B)$, as the supremum of $\tilde{S}$ and the Jordan outer measure, $m^{*,(J)}(B)$, as the infimum of S. Or, more formally:

Definition 2.6 (Jordan measure)**.** Suppose $E \subset \mathbb{R}^d$ is a bounded subset. We have the following:

a) The *Jordan inner measure*, denoted by $m_{*,(J)}(E)$ is defined as:

$$m_{*,(J)}(E) := \sup_{A\subset E, A \text{ elementary}} m(A)$$

b) The *Jordan outer measure*, denoted by $m^{*,(J)}(E) := \inf_{B \supset E, B \text{ elementary}} m(B)$
c) If $m_{*,(J)}(E) = m^{*,(J)}(E)$, then we define E as *Jordan measurable*. We call $m(E) := m_{*,(J)}(E) = m^{*,(J)}(E)$ the *Jordan measure* of E and denote this by $m_{(J)}(E)$.

From this, we can deduce the following equivalent statements for any bounded subset $E \in \mathbb{R}^d$:

a) E is Jordan measurable
b) For every $\epsilon > 0$, elementary sets $A \subset E \subset B$ exist such that $m(B \backslash A) \leq \epsilon$.
c) For every $\epsilon > 0$, an elementary set A exists such that $m^{*,(J)}(A \Delta E) \leq \epsilon$.

Interestingly enough, there is a connection between the Jordan measure and Riemann integration. Naturally, since we have the measure of a set computing the "size" of it, Riemann integration provides ample support to measure certain geometries.

3. Darboux and Riemann Integration

Riemann integration is commonly known to be the very first rigorous form of integration, and for the sake of this paper, we will be covering the 1-dimensional Riemann integral (although higher-dimensional analogues can be constructed through Lebesgue integration). An important concept in Riemann integration is the *tagged partition*, which informally, is a partition that links intervals with points. Or,

Definition 3.1 (Tagged Partition)**.** Suppose we have a positive interval $[a, b]$ with a function $f : [a, b] \to \mathbb{R}$. A *tagged partition* $\mathcal{P} = ((x_0, ..., x_n), (\bar{x_1}, ..., \bar{x_n}))$ of $[a, b]$ is a finite sequence of real numbers $a = x_0 < ... < x_n = b$ with $x_{i-1} \leq \bar{x_i} \leq x_i$ for each $i = 1, ..., n$. We rewrite $x_i - x_{i-1}$ as δx_i.

Definition 3.2 (Riemann Sum)**.** The *Riemann Sum* $\mathcal{R}(f, \mathcal{P}$ of a function $f : [a, b] \to \mathbb{R}$ with respect to the tagged partition $\mathcal{P}$ is

$$\mathcal{R}(f, \mathcal{P}) := \sum_{i=1}^{n} f(\bar{x_i} \delta x_i).$$

f is *Riemann integrable* on $[a, b]$ if

$$\int_a^b f(x)\, dx = \lim_{\Delta(\mathcal{P}) \to 0} \mathcal{R}(f, \mathcal{P}).$$

This idea leads into the following definitions:

Definition 3.3 (Darboux Integral)**.** Suppose we have an interval $[a, b]$ and a bounded function $f : [a, b] \to \mathbb{R}$. The *lower Darboux integral* of f on $[a, b]$ is:

$$\int_a^b f(x)\, dx := \sup_{g \leq f, \text{p.c.}} \text{p.c.} \int_a^b g(x)\, dx,$$

where p.c. is the piecewise constant and g is ranged over all p.c. functions that are bounded by f.

We define the *upper Darboux integral* as the same except with the infimum instead of the supremum. If both the upper and lower Darboux integrals are the same, we say that f is Darboux integrable. Now, we have enough to draw connections between the Jordan measure and the Riemann integral.

Theorem 3.4. *Suppose we have an integral $[a, b]$ and two Riemann integrable functions $f, g : [a, b] \to \mathbb{R}$. Then, the following hold.*

(1) For any $c \in \mathbb{R}$, cf and $f + g$ are Riemann integrable and $\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$ and $\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$.

(2) If $f(x) \leq g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$.

(3) If E is a Jordan measurable on $[a, b]$, then the indicator function $1_E : [a, b] \to \mathbb{R}$ is Riemann integrable and $\int_a^b 1_E(x)\, dx = m(E)$.

These properties end up uniquely identify the properties of the Riemann integral. This means that the map $f \mapsto \int_a^b f(x)\, dx$ is the *only* mapping that follows the above 3 properties. We can also interpret this in terms of "area under a curve" with the following theorem:

Theorem 3.5. *Suppose we have a bounded function $f : [a, b] \to \mathbb{R}$. If we have Jordan measurable sets $E^* := \{(x, t) : x \in [a, b],\ 0 \leq t \leq f(x)\}$ and $E_* := \{(x, t) : x \in [a, b],\ f(x) \leq t \leq 0\}$ in $\mathbb{R}^2$, then*

$$\int_a^b f(x)\, dx = m^2(E^*) - m^2(E_*)$$

where m^2 is the 2-dimensional Jordan measure.

This completes our connection between the Jordan measure and the Riemann integral.

References

[1] Frank Porter. The lebesgue measure.
[2] Terence Tao. An introduction to measure theory. 2011.